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Expansion of cylindrical shock waves with counterpressure produced by instantaneous release of energy

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Abstract

In this thesis, the expansion of cylindrical shock waves produced by instantaneous release of energy per unit length along a line is investigated, as it can occur due to a line explosion or a spark discharge. First, the conditions of a strong shock wave are presumed, i.e. the pressure of the surrounding is – compared to the pressure behind the shock – very low and thus can be neglected. Afterwards the equations are expanded to consider and calculate the impact of the counter-pressure. Since those equations are complex, they are simplified by asymptotic expansion for small times and only the leading and first order terms are considered. The numerical implementation of this problem is described, and all obtained results are ploted and discussed, and conclusions are given.

Nomenclature

Superscripts

$\sim$	dimensional
$'$	partial derivate with respect to η
$\cdot$	partial derivate with respect to τ
h	homogeneous part of the differential equation system
p	particular part of the differential equation system
(m)	discretisation step with respect to τ
(n)	discretisation step with respect to η

Subscripts

0	order $\mathcal{O}(\tau^0)$
1	order $\mathcal{O}(\tau^1)$
s	immediately behind the shock wave
u	undisturbed surrounding

Symbols

$\Delta\eta$	step size
η	radius scaling function
γ	ratio of specific heats
ϕ	velocity scaling function

Π	non-dimensional product to determine characteristic values
ψ	density scaling function
$\tilde{\rho}$	density
τ	time scaling function
$\tilde{A}$	parameter dependent on τ; constant
$\tilde{a}$	speed of sound
$\tilde{c}_v$	specifit heat capacity at constant volume
$\tilde{e}$	internal energy
$\tilde{E}_c$	characteristic energy per unit length
$\tilde{E}_{cyl}$	energy per unit length inside the cylinder
$\tilde{E}_{tot}$	total energy per unit length in the system
$\tilde{E}_u$	energy per unit length as pressure-volume work
f	pressure scaling function
$\tilde{l}_c$	characteristic length
$\tilde{M}$	molecular weight of air
$\tilde{m}$	mass per unit length
Ma	Mach number
$\tilde{P}$	perimeter of the shock wave cylinder
$\tilde{p}$	pressure
$\tilde{R}$	radius of shock wave
$\tilde{r}$	radius
$\tilde{R}_a$	common gas constant for air
$\tilde{R}_u$	universal gas constant

$\tilde{T}$	temperature
$\tilde{t}$	time
$\tilde{t}_c$	characteristic time
$\tilde{T}_{ref}$	reference temperature to define the internal energy
$\tilde{U}$	speed of shock wave
$\tilde{u}$	velocity

1 Introduction

The instantaneous release of energy along a straight line is investiaged as it happens in the first stage of an explosion or in spark discharge. Afterwards the expansion of the resulting shock wave is described and calculated. Figure 1.1 shows the shock wave cylinder for two different times.

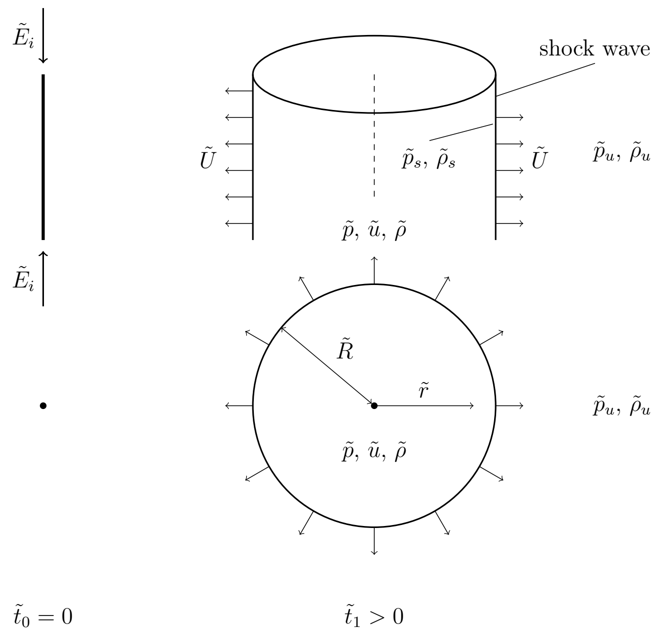


Figure 1.1: Model of the cylindrical shock wave.

The results of this thesis will be embedded and used in a research project [3] [4], where the ignition in gas engines will be modeled. The aim of this thesis is to provide a profile of pressure, density and velocity at a certain time.

Lin [10] extended G.I. Talylor's analysis [2] of strong spherical explosions to strong cylindrical shock waves, presented a set of ordinary differential equations for a similarity solution and solved it nummerically. Here, we follow the scaling of [10] and consider the case of a non-vanishing counter-pressure.

Plooster [7] investigated this problem, too. The main difference between his approach and the one used in this thesis is Plooster's definition of the fluid: he does not consider it as an ideal gas and uses an equation of state called Saha ionization equation which considers the dissociation and ionization of the fluid. Furthermore, he does not scale time and radius, respectively. Therefore, no similarity solution exists and the results are purely numerical.

Ö. Ekici et al. [8] present a model for fast spark discharge with focus on the spark discharge mechanism. However, we will focus on the fluid mechanical aspects.

In chapter 2,the governing equations will be introduced. In section 2.1, the problem will be simplified in order to get a preliminary solution. Section 2.2 will provide us with a more general approach to solving this problem. First, a system of partial differential equations will be solved, and with the expansion made in section 2.2.1, said system will become a system of ordinary differential equations. The boundary conditions described in section 2.2.2 are set by jump conditions for pressure, density and velocity. Some parameters we introduce can be calculated with an energy balance, as described in section 2.2.3.

In chapter 3, the numerical implementation of the equations previously obtained will be presented. The impact of the number of steps used for calculations and its effect on the precision of the results is investigated.

In chapter 4, a numerical solution with chosen parameters will be showed. In section 4.1, the limitations of the solutions will be discussed. Any numerical solution needs to correspond with the general laws of physics, these restrictions then limit the possible range of certain parameters. In section 4.2, the numerical solutions are presented and plotted.

In chapter 5, the solution obtained is discussed and some conclusions are given.

2 Physical model

In order to solve the problem presented in chapter 1, the following will be assumed [10]:

- constant specific heat capacities,
- friction and heat conduction and radiation are neglected.
- The problem given is considered to be cylindrically symmetrical.

The fundamental equations for solving the given problem are the Euler equation of momentum and the equation of continuity. With time $\tilde{t}$, radius $\tilde{r}$, pressure $\tilde{p}$, velocity $\tilde{u}$ and density $\tilde{\rho}$, it reads:

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\,\frac{\partial \tilde{u}}{\partial \tilde{r}} = -\frac{1}{\tilde{\rho}}\,\frac{\partial \tilde{p}}{\partial \tilde{r}}, \tag{2.1}$$

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \tilde{u}\,\frac{\partial \tilde{\rho}}{\partial \tilde{r}} + \tilde{\rho}\left(\frac{\partial \tilde{u}}{\partial \tilde{r}} + \frac{\tilde{u}}{\tilde{r}}\right) = 0. \tag{2.2}$$

The total derivate (derivate in frame of reference of fluid particle) of the entropy $\tilde{s}$ is constant as long as the particle is not hit by a shock [9], where γ is the ratio of specific heats:

$$\left(\frac{\partial}{\partial \tilde{t}} + \tilde{u}\,\frac{\partial}{\partial \tilde{r}}\right)\left(\frac{\tilde{p}}{\tilde{\rho}^{\gamma}}\right) = 0. \tag{2.3}$$

Furthermore, the shock front is located at $\tilde{r} = \tilde{R}$. Therefore, the solutions to equations (2.1) to (2.3) must fulfill the jump conditions over the shock wave which follow from [5] and [10]. With the speed of the shock $\tilde{U}$, and the entities before and after the shock – subscript u and s, respectively – the following equations are obtained:

$$\frac{\tilde{p}_s}{\tilde{p}_u} = \frac{2\,\gamma}{\gamma+1}\,\mathrm{Ma}^2 - \frac{\gamma-1}{\gamma+1}, \tag{2.4}$$

$$\frac{\tilde{\rho}_s}{\tilde{\rho}_u} = \frac{\gamma+1}{\gamma-1+2\,\mathrm{Ma}^{-2}}, \tag{2.5}$$

$$\frac{\tilde{u}_s}{\tilde{U}} = \frac{2}{\gamma+1}\left(1-\mathrm{Ma}^{-2}\right), \tag{2.6}$$

$$\tilde{U} = \frac{\mathrm{d}\tilde{R}}{\mathrm{d}\tilde{t}}, \tag{2.7}$$

with the speed of sound $\tilde{a}$ and the Mach number Ma

$$\tilde{a} = \sqrt{\frac{\gamma\,\tilde{p}_u}{\tilde{\rho}_u}}, \tag{2.8}$$

$$\mathrm{Ma} = \frac{\tilde{U}}{\tilde{a}}. \tag{2.9}$$

For the formulation of the energy balance, a system whose boundaries are just outside of the shock wave is chosen. At this boundary, the gas is still at rest and thus, the increase of energy per unity time of the control volume is equal to the internal energy of the mass which is passed over with the shock wave moving boundary, see figure 2.1.

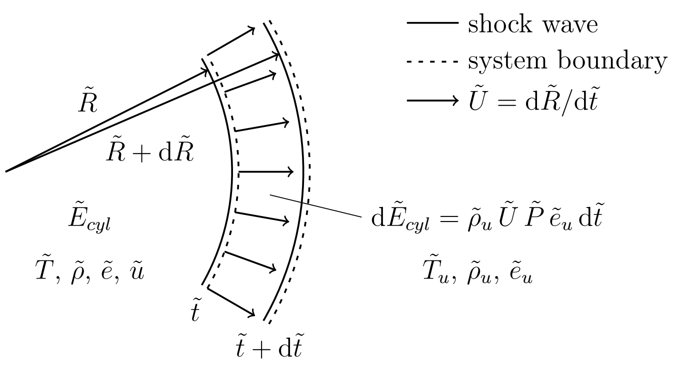


Figure 2.1: System boundary and energy per unit length transfer into the system.

$$\mathrm{d}\tilde{E}_{tot} = \tilde{\rho}_u\,\tilde{U}\,\tilde{P}\,\tilde{e}_u\,\mathrm{d}\tilde{t}, \tag{2.10}$$

with the perimeter of the chosen system $\tilde{P}$, specific heat at constant volume $\tilde{c}_v$ and the definition of internal energy per unit length $\tilde{e}$

$$\tilde{e}(\tilde{T}) = \tilde{c}_v\,(\tilde{T} - \tilde{T}_{ref}), \tag{2.11}$$

and thus

$$\frac{\mathrm{d}\tilde{E}_{tot}}{\mathrm{d}\tilde{t}} = \tilde{\rho}_u\,\tilde{U}\,\tilde{P}\,\tilde{c}_v\,\left(\tilde{T}_u - \tilde{T}_{ref}\right). \tag{2.12}$$

The energy per unit length in the system $\tilde{E}_{cyl}$ consists of the internal energy per unit length and the kinetic energy per unit length in the system. By integrating over the area of the cross section of the shock wave cylinder, $\tilde{E}_{cyl}$ conforms with:

$$\tilde{E}_{cyl} = \int\limits_{\tilde{A}} \left(\tilde{\rho}\,\tilde{e} + \frac{1}{2}\,\tilde{\rho}\,\tilde{u}^2\right)\mathrm{d}\tilde{A} = \int\limits_{\tilde{A}} \left(\tilde{\rho}\,\tilde{c}_v\,(\tilde{T} - \tilde{T}_{ref}) + \frac{1}{2}\,\tilde{\rho}\,\tilde{u}^2\right)\mathrm{d}\tilde{A}. \tag{2.13}$$

By setting the reference temperature of internal energy $\tilde{T}_{ref}$ to $\tilde{T}_u$, the energy per unit length in the system remains constant, see equation (2.12). Therfore, the total energy per unit length $\tilde{E}_{tot}$ can be defined:

$$\tilde{E}_{tot} = 2\,\pi \int_0^{\tilde{R}} \tilde{\rho}\left(\tilde{c}_v\left(\tilde{T} - \tilde{T}_u\right) + \frac{1}{2}\,\tilde{u}^2\right)\,\tilde{r}\,\mathrm{d}\tilde{r}, \tag{2.14}$$

or, using the ideal gas equation (see also [1]):

$$\tilde{E}_{tot} = 2\,\pi \int_0^{\tilde{R}} \left(\frac{\tilde{p} - \tilde{p}_u}{\gamma - 1} + \frac{1}{2}\,\tilde{\rho}\,\tilde{u}^2\right)\,\tilde{r}\,\mathrm{d}\tilde{r}. \tag{2.15}$$

2.1 Solution for strong shock waves

First, it is assumed that the pressure in front of the shock wave has no influence on the shock or the fluid behind it, i.e. the pressure after the shock wave is significantly higher than the pressure before, $\tilde{p}_s \gg \tilde{p}_u$, or, in other words, the Mach number is high, Ma $\gg$ 1. Because of lack of a reference length and time, it is possible to introduce a similarity varible η and the partial differential equations (2.1) to (2.3) become ordinary differential equations. By scaling pressure, density and velocity the solutions obtained are non-dimensional. First, partially dimensional scaling functions are introduced. Later, another set of completely non-dimensional scaling functions is used. Introducing the dimensional scaling functions for pressure $\tilde{f}$ and velocity $\tilde{\phi}$, the non-dimensional scaling function for density $\tilde{\psi}$, and the similarity variables η which scales radius $\tilde{r}$ and τ which scales time $\tilde{t}$ [10]:

$$\tilde{p}(\tilde{r},\tilde{t}) = \frac{\tilde{p}_u}{\tilde{R}^2(\tau)}\,\tilde{f}(\eta), \tag{2.16}$$

$$\tilde{u}(\tilde{r},\tilde{t}) = \frac{1}{\tilde{R}(\tau)}\,\tilde{\phi}(\eta), \tag{2.17}$$

$$\tilde{\rho}(\tilde{r},\tilde{t}) = \tilde{\rho}_u\,\psi(\eta), \tag{2.18}$$

$$\tilde{r} = \tilde{R}(\tau)\,\eta(\tilde{r},\tilde{t}), \tag{2.19}$$

$$\tilde{t} = \tilde{t}_c\,\tau(\tilde{t}). \tag{2.20}$$

Reference time $\tilde{t}_c$ and reference length $\tilde{l}_c$ are introduced here and will be defined in section 2.2 when the counter-pressure is reintroduced into the problem. Inserting the scaling functions into equations (2.1) to (2.3), following equations are obtained after rearranging terms and putting the equations in standard variable separated form:

$$\frac{\tilde{R}\,\dot{\tilde{R}}}{\tilde{t}_c} = \left(\tilde{\phi}\,\tilde{\phi}' + \frac{\tilde{p}_u}{\tilde{\rho}_u}\frac{\tilde{f}'}{\psi}\right)\left(\tilde{\phi} + \eta\,\tilde{\phi}'\right)^{-1}, \tag{2.21}$$

$$\frac{\tilde{R}\,\dot{\tilde{R}}}{\tilde{t}_c} = \frac{\tilde{\phi}}{\eta} + \left(\tilde{\phi}' + \frac{\tilde{\phi}}{\eta}\right)\frac{\psi}{\eta\,\psi'}, \tag{2.22}$$

$$\frac{\tilde{R}\,\dot{\tilde{R}}}{\tilde{t}_c} = \left(\gamma\,\tilde{f}\,\tilde{\phi}\,\frac{\psi'}{\psi} - \tilde{\phi}\,\tilde{f}'\right)\left(\gamma\,\eta\,\tilde{f}\,\frac{\psi'}{\psi} - \eta\,\tilde{f}' - 2\,\tilde{f}\right)^{-1}. \tag{2.23}$$

Given that the left part of equations (2.21) to (2.23) is only dependent on time, whereas the right part is only a funtion of η, both parts must be constant. The resulting constant is being called $\tilde{A}$:

$$\tilde{A} := \frac{\tilde{R}\,\dot{\tilde{R}}}{\tilde{t}_c}. \tag{2.24}$$

The time independent constant $\tilde{A}$ has the dimension of a velocity multiplied by a length. With this definition, it is possible to introduce a set of completely non-dimensional scaling functions [10]:

$$\tilde{f}(\eta) = \frac{\tilde{A}^2}{\tilde{a}^2}\, f(\eta), \tag{2.25}$$

$$\tilde{\phi}(\eta) = \tilde{A}\,\phi(\eta). \tag{2.26}$$

With these new functions, equations (2.21) to (2.23) become, after sorting the individual terms with respect to the derivate:

$$\gamma^{-1} f' - (\eta - \phi)\,\psi\,\phi' = \phi\,\psi, \tag{2.27}$$

$$-\psi\,\phi' + (\eta - \phi)\,\psi' = \phi\,\psi\,\eta^{-1}, \tag{2.28}$$

$$-(\eta - \phi)\,\psi\, f' + \gamma\,(\eta - \phi)\, f\,\psi' = 2\, f\,\psi. \tag{2.29}$$

In order to describe the problem more clearly, the system of ordinary differential equations is written in matrix-vector form:

$$\boldsymbol{L}\,\boldsymbol{x}' = \boldsymbol{n} \tag{2.30}$$

with

$$\boldsymbol{L} = \begin{pmatrix} \gamma^{-1} & -(\eta - \phi)\,\psi & 0 \\ 0 & -\psi & \eta - \phi \\ -(\eta - \phi)\,\psi & 0 & \gamma\, f\,(\eta - \phi) \end{pmatrix},$$

$$\boldsymbol{x} = \begin{pmatrix} f(\eta) & \phi(\eta) & \psi(\eta) \end{pmatrix}^{\mathsf{T}},$$

$$\boldsymbol{n} = \begin{pmatrix} \phi\,\psi & \phi\,\psi\,\eta^{-1} & 2\, f\,\psi \end{pmatrix}^{\mathsf{T}}.$$

This system of ordinary differential equations can be solved subjected to the boundary conditions given by equations (2.4) to (2.6). Inserting the scaling functions into the boundary conditions, it reads as follows:

$$f(1) = \frac{2\,\gamma}{\gamma+1} - \frac{\gamma-1}{\gamma+1}\,\mathrm{Ma}^{-2}, \tag{2.31}$$

$$\phi(1) = \frac{2}{\gamma+1} - \frac{2}{\gamma+1}\,\mathrm{Ma}^{-2}, \tag{2.32}$$

$$\psi(1) = \frac{\gamma+1}{\gamma-1} - \frac{2\,(\gamma+1)}{(\gamma-1)^2}\,\mathrm{Ma}^{-2}. \tag{2.33}$$

For the similarity solution a very high Mach number is assumed, $\mathrm{Ma} \gg 1$. Therefore, the boundary conditions simplify to:

$$f(1) \sim \frac{2\,\gamma}{\gamma+1}, \tag{2.34}$$

$$\phi(1) \sim \frac{2}{\gamma+1}, \tag{2.35}$$

$$\psi(1) \sim \frac{\gamma+1}{\gamma-1} \tag{2.36}$$

The imparted energy per unit length must remain constant in the system, therefore $\tilde{A}$ can be calculated by inserting the scaling functions into equation (2.15), with the assumption that $\tilde{p}_s \gg \tilde{p}_u$:

$$\tilde{E}_{tot} = 2\,\pi\,\tilde{\rho}_u\,\tilde{A}^2 \int_0^1 \left(\frac{f}{\gamma\,(\gamma-1)} + \frac{\phi^2\psi}{2} \right) \eta\,\mathrm{d}\eta. \tag{2.37}$$

2.2 Extension to shock waves with counter-pressure

For this extension, the counter-pressure in front of the shock wave is not neglected any longer. Therefore, the scaling functions (2.16) to (2.18) change and are now explicitly time, i.e. τ, dependent as follows:

$$\tilde{p}(\tilde{r},\tilde{t}) = \frac{\tilde{p}_u}{\tilde{R}^2(\tau)}\,\tilde{f}(\eta,\tau), \tag{2.38}$$

$$\tilde{u}(\tilde{r},\tilde{t}) = \frac{1}{\tilde{R}(\tau)}\,\tilde{\phi}(\eta,\tau), \tag{2.39}$$

$$\tilde{\rho}(\tilde{r},\tilde{t}) = \tilde{\rho}_u\,\psi(\eta,\tau). \tag{2.40}$$

Because of the time dependence in the scaling functions $\tilde{f}$, $\tilde{\phi}$ and ψ, there are additional terms in equations (2.21) to (2.23):

$$\frac{\tilde{R}\,\dot{\tilde{R}}}{\tilde{t}_c} = \left(\tilde{\phi}\,\tilde{\phi}' + \frac{\tilde{p}_u}{\tilde{\rho}_u}\,\frac{\tilde{f}'}{\psi} + \frac{\tilde{R}^2}{t_c}\,\dot{\tilde{\phi}}\right)\left(\tilde{\phi} + \eta\,\tilde{\phi}'\right)^{-1}, \tag{2.41}$$

$$\frac{\tilde{R}\,\dot{\tilde{R}}}{\tilde{t}_c} = \frac{\tilde{\phi}}{\eta} + \left(\tilde{\phi}' + \frac{\tilde{\phi}}{\eta}\right)\frac{\psi}{\eta\,\psi'} + \frac{\tilde{R}^2}{\eta\,t_c}\,\frac{\dot{\psi}}{\psi'}, \tag{2.42}$$

$$\frac{\tilde{R}\,\dot{\tilde{R}}}{\tilde{t}_c} = \left(\gamma\,\tilde{f}\,\tilde{\phi}\,\frac{\psi'}{\psi} - \tilde{\phi}\,\tilde{f}' + \frac{\tilde{R}^2}{t_c}\left(\frac{\gamma\,f\,\dot{\psi}}{\psi} - \dot{\tilde{f}}\right)\right)\left(\gamma\,\eta\,\tilde{f}\,\frac{\psi'}{\psi} - \eta\,\tilde{f}' - 2\,\tilde{f}\right)^{-1}. \tag{2.43}$$

The left part of equations (2.41) to (2.43) is only dependent on time, whereas the right part is merely a function of η and τ. Thus, parameter $\tilde{A}$ can only depend on τ:

$$\tilde{A}(\tau) := \frac{\tilde{R}(\tau)\,\dot{\tilde{R}}(\tau)}{\tilde{t}_c}. \tag{2.44}$$

With this definition, the completely non-dimensional scaling functions conform with:

$$\tilde{f}(\eta,\tau) = \frac{\tilde{A}^2(\tau)}{\tilde{a}^2}\,f(\eta,\tau), \tag{2.45}$$

$$\tilde{\phi}(\eta,\tau) = \tilde{A}(\tau)\,\phi(\eta,\tau), \tag{2.46}$$

$$\tilde{A}(\tau) = \frac{\tilde{l}_c^2}{\tilde{t}_c}\,A(\tau), \tag{2.47}$$

$$\tilde{R}(\tau) = \tilde{l}_c\,R(\tau). \tag{2.48}$$

Inserting the scaling functions into equations (2.41) to (2.43) and ordering terms with respect to the derivates, it follows:

$$\gamma^{-1}\,f' - (\eta - \phi)\psi\,\phi' = \phi\,\psi - \frac{R^2}{A}\,\psi\,\dot{\phi} - \frac{R^2}{A^2}\,\psi\,\phi\,\dot{A}, \tag{2.49}$$

$$-\psi\,\phi' + (\eta-\phi)\psi' = \phi\,\psi\,\eta^{-1} + \frac{R^2}{A}\,\dot{\psi}, \tag{2.50}$$

$$-(\eta-\phi)\,\psi\,f' + \gamma\,f(\eta-\phi)\,\psi' = 2\,f\,\psi + \frac{R^2}{A}\left(\gamma\,f\,\dot{\psi} - \psi\,\dot{f}\right) - 2\,\frac{R^2}{A^2}\,f\,\psi\,\dot{A}. \tag{2.51}$$

Putting this system of differential equations in matrix form, it reads as follows:

$$\boldsymbol{L}\,\boldsymbol{x}' = \boldsymbol{n} + \frac{R^2}{A}\,\hat{\boldsymbol{K}}\,\dot{\boldsymbol{x}} + \frac{R^2}{A^2}\,\hat{\boldsymbol{m}}\,\dot{A}, \tag{2.52}$$

with

$$\begin{aligned}
\boldsymbol{L} &= \begin{pmatrix} \gamma^{-1} & -(\eta-\phi)\,\psi & 0 \\ 0 & -\psi & \eta-\phi \\ -(\eta-\phi)\,\psi & 0 & \gamma\,f\,(\eta-\phi) \end{pmatrix}, \\
\boldsymbol{x} &= \begin{pmatrix} f(\eta,\tau) & \psi(\eta,\tau) & \psi(\eta,\tau) \end{pmatrix}^{\mathsf{T}}, \\
\boldsymbol{n} &= \begin{pmatrix} \phi\,\psi & \phi\,\psi\,\eta^{-1} & 2\,f\,\psi \end{pmatrix}^{\mathsf{T}}, \\
\hat{\boldsymbol{K}} &= \begin{pmatrix} 0 & -\psi & 0 \\ 0 & 0 & 1 \\ -\psi & 0 & \gamma\,f \end{pmatrix}, \\
\hat{\boldsymbol{m}} &= \begin{pmatrix} -\phi\,\psi & 0 & -2\,f\,\psi \end{pmatrix}^{\mathsf{T}}.
\end{aligned}$$

2.2.1 Expansion with respect to small τ

Now all functions and variables are expanded with respect to τ. Entities of oder $\mathcal{O}(\tau^0)$ have subscript 0, entities of order $\mathcal{O}(\tau^1)$ have subsrcipt 1:

$$f(\eta,\tau) \sim f_0(\eta) + \tau\,f_1(\eta), \tag{2.53}$$

$$\phi(\eta,\tau) \sim \phi_0(\eta) + \tau\,\phi_1(\eta), \tag{2.54}$$

$$\psi(\eta,\tau) \sim \psi_0(\eta) + \tau\,\psi_1(\eta), \tag{2.55}$$

$$R(\tau) \sim \sqrt{\tau}\,(R_0 + \tau\,R_1)\,, \tag{2.56}$$

$$A(\tau) \sim A_0 + \tau\,A_1. \tag{2.57}$$

Now two different systems of equations have to be solved: one with order $\mathcal{O}(\tau^0)$, and another one with order $\mathcal{O}(\tau^1)$. Expanding equation (2.52) with respect to the order of τ will become for the order $\mathcal{O}(\tau^0)$:

$$\boldsymbol{L}_0\, \boldsymbol{x}_0' = \boldsymbol{n}_0, \tag{2.58}$$

with

$$\begin{aligned}
\boldsymbol{L}_0 &= \begin{pmatrix} \gamma^{-1} & -(\eta-\phi_0)\,\psi_0 & 0 \\ 0 & -\psi_0 & \eta-\phi_0 \\ -(\eta-\phi_0)\,\psi_0 & 0 & \gamma\, f_0\,(\eta-\phi_0) \end{pmatrix}, \\
\boldsymbol{n}_0 &= \begin{pmatrix} \phi_0\,\psi_0 & \phi_0\,\psi_0\,\eta^{-1} & 2\, f_0\,\psi_0 \end{pmatrix}^{\intercal},
\end{aligned}$$

which equals the system described in section 2.1, and for the order $\mathcal{O}(\tau^1)$:

$$\boldsymbol{L}_0\, \boldsymbol{x}_1' = -\boldsymbol{L}_1\, \boldsymbol{x}_0' + \boldsymbol{n}_1 + \frac{R_0^2}{A_0}\, \hat{\boldsymbol{K}}_0\, \boldsymbol{x}_1 + \frac{R_0^2}{A_0^2}\, \hat{\boldsymbol{m}}_0\, A_1, \tag{2.59}$$

with

$$R_0^2 = 2\, A_0, \tag{2.60}$$

and, thus

$$\boldsymbol{L}_0\, \boldsymbol{x}_1' = -\boldsymbol{L}_1\, \boldsymbol{x}_0' + \boldsymbol{n}_1 + \boldsymbol{K}_0\, \boldsymbol{x}_1 + \boldsymbol{m}_0\, A_1, \tag{2.61}$$

with

$$\boldsymbol{L}_1 = \begin{pmatrix} 0 & \phi_1\,\psi_0 - (\eta-\phi_0)\,\psi_1 & 0 \\ 0 & -\psi_1 & -\phi_1 \\ \phi_1\,\psi_0 - (\eta-\phi_0)\,\psi_1 & 0 & \gamma\,(f_1\,(\eta-\phi_0) - f_0\,\phi_1) \end{pmatrix},$$

$$\boldsymbol{n}_1 = \left(\phi_1\,\psi_0 + \phi_0\,\psi_1 \quad (\phi_1\,\psi_0 + \phi_0\,\psi_1)\,\eta^{-1} \quad 2\,(f_1\,\psi_0 + f_0\,\psi_1)\right)^\mathsf{T},$$

$$\boldsymbol{K}_0 = \begin{pmatrix} 0 & -2\,\psi_0 & 0 \\ 0 & 0 & 2 \\ -2\,\psi_0 & 0 & 2\,\gamma\,f_0 \end{pmatrix},$$

$$\boldsymbol{m}_0 = \left(-2\,\phi_0\,\psi_0\,A_0^{-1} \quad 0 \quad -4\,f_0\,\psi_0\,A_0^{-1}\right)^\mathsf{T}.$$

In order to simplify the structure of this equation system, $\boldsymbol{x_1}$ is singled out:

$$\boldsymbol{L}_0\,\boldsymbol{x}_1' = -\boldsymbol{L}_1\,\boldsymbol{x}_0' + \boldsymbol{n}_1 + \boldsymbol{K}_0\,\boldsymbol{x}_1 + \boldsymbol{m}_0\,A_1, \tag{2.62}$$

$$\hat{\boldsymbol{L}}_0\,\boldsymbol{x}_1 := -\boldsymbol{L}_1\,\boldsymbol{x}_0', \tag{2.63}$$

$$\boldsymbol{N}_0\,\boldsymbol{x}_1 := \boldsymbol{n}_1, \tag{2.64}$$

$$\boldsymbol{L}_0\,\boldsymbol{x}_1' = \left(\hat{\boldsymbol{L}}_0 + \boldsymbol{N}_0 + \boldsymbol{K}_0\right)\,\boldsymbol{x}_1 + \boldsymbol{m}_0\,A_1, \tag{2.65}$$

$$\boldsymbol{P}_0 := \hat{\boldsymbol{L}}_0 + \boldsymbol{N}_0 + \boldsymbol{K}_0, \tag{2.66}$$

with

$$\hat{\boldsymbol{L}}_0 = \begin{pmatrix} 0 & -\phi_0'\,\psi_0 & (\eta - \phi_0)\,\phi_0' \\ 0 & \psi_0' & \phi_0' \\ -\gamma\,(\eta - \phi_0)\,\psi_0' & \gamma\,f_0\,\psi_0' - f_0'\,\psi_0 & (\eta - \phi_0)\,f_0' \end{pmatrix},$$

$$\boldsymbol{N}_0 = \begin{pmatrix} 0 & \psi_0 & \phi_0 \\ 0 & \psi_0\,\eta^{-1} & \phi_0\,\eta^{-1} \\ 2\,\psi_0 & 0 & 2\,f_0 \end{pmatrix}.$$

Now the equation system in matrix form is more structured:

$$\boldsymbol{L}_0\,\boldsymbol{x}_1' = \boldsymbol{P}_0\,\boldsymbol{x}_1 + \boldsymbol{m}_0\,A_1, \tag{2.67}$$

with

$$\boldsymbol{P}_0 = \begin{pmatrix} 0 & -\psi_0 - \phi_0'\,\psi_0 & (\eta - \phi_0)\,\phi_0' + \phi_0 \\ 0 & \psi_0\,\eta^{-1} + \psi_0' & \phi_0\,\eta^{-1} + \phi_0' + 2 \\ -\gamma\,(\eta - \phi_0)\,\psi_0' & \gamma\,f_0\,\psi_0' - f_0'\,\psi_0 & (\eta - \phi_0)\,f_0' + 2\,f_0\,(1+\gamma) \end{pmatrix}.$$

2.2.2 Boundary conditions

The boundary conditions dependent on the Mach number Ma were given by equations (2.31) to (2.33). In order to be able to expand those equations, a correlcation of Ma^{-2} is necessary. The lowest order of Ma^{-2} is $\mathcal{O}(\tau^1)$:

$$\mathrm{Ma}^{-2} = \left(\frac{\tilde{a}}{\mathrm{d}\tilde{R}/\mathrm{d}\tilde{t}}\right)^2, \tag{2.68}$$

$$\frac{\mathrm{d}\tilde{R}}{\mathrm{d}\tilde{t}} = \frac{\mathrm{d}R}{\mathrm{d}\tau}\,\frac{\tilde{l}_c}{\tilde{t}_c}, \tag{2.69}$$

$$\mathrm{Ma}^{-2} \sim \frac{4\,\tilde{a}^2\,\tilde{t}_c^2}{R_0^2\,\tilde{l}_c^2}\,\tau. \tag{2.70}$$

Therefore, the boundary conditions conform with:

$$f(1,\tau) \sim f_0(1) + \tau\,f_1(1), \tag{2.71}$$

$$\phi(1,\tau) \sim \phi_0(1) + \tau\,\phi_1(1), \tag{2.72}$$

$$\psi(1,\tau) \sim \psi_0(1) + \tau\,\psi_1(1), \tag{2.73}$$

for the order $\mathcal{O}(\tau^0)$:

$$f_0(1) = \frac{2\,\gamma}{\gamma+1}, \tag{2.74}$$

$$\phi_0(1) = \frac{2}{\gamma+1}, \tag{2.75}$$

$$\psi_0(1) = \frac{\gamma+1}{\gamma-1}, \tag{2.76}$$

and for the order $\mathcal{O}(\tau^1)$:

$$f_1(1) = -\frac{\gamma-1}{\gamma+1}\frac{4\,\tilde{a}^2\,\tilde{t}_c^2}{R_0^2\,\tilde{l}_c^2}, \tag{2.77}$$

$$\phi_1(1) = -\frac{2}{\gamma+1}\frac{4\,\tilde{a}^2\,\tilde{t}_c^2}{R_0^2\,\tilde{l}_c^2}, \tag{2.78}$$

$$\psi_1(1) = -\frac{2(\gamma+1)}{(\gamma-1)^2}\frac{4\,\tilde{a}^2\,t_c^2}{R_0^2\,\tilde{l}_c^2}. \tag{2.79}$$

Setting $\tilde{a}\,\tilde{t}_c/\tilde{l}_c = 1$ (see also section 2.2.3), the boundary conditions in new scaling for $\eta = 1$ read as follows:

$$f_1(1) = -\frac{2\,(\gamma-1)}{\gamma+1}\,A_0^{-1}, \tag{2.80}$$

$$\phi_1(1) = -\frac{4}{\gamma+1}\,A_0^{-1}, \tag{2.81}$$

$$\psi_1(1) = -\frac{4\,(\gamma+1)}{(\gamma-1)^2}\,A_0^{-1}. \tag{2.82}$$

2.2.3 Energy balance

The energy per unit length inside the shock wave cylinder consists of internal energy per unit length and kinetic energy per unit length. By integrating over the area, $\tilde{E}_{cyl}$ equals:

$$\tilde{E}_{cyl} = 2\,\pi \int_0^{\tilde{R}} \left(\frac{\tilde{p}}{\gamma-1} + \frac{1}{2}\,\tilde{\rho}\,\tilde{u}^2\right)\,\tilde{r}\,\mathrm{d}\tilde{r}. \tag{2.83}$$

Since the counter-pressure in front of the shock wave is not neglected anymore, a definition of an energy per unit length $\tilde{E}_u$ that reflects the counter-pressure can be defined as:

$$\tilde{E}_u = -2\,\pi \int_0^{\tilde{R}} \frac{\tilde{p}_u}{\gamma-1}\,\tilde{r}\,\mathrm{d}\tilde{r}. \tag{2.84}$$

The total energy per unit length $\tilde{E}_{tot}$ in the system is composed of the energy per unit length in the cylinder $\tilde{E}_{cyl}$, and the energy per unit length $\tilde{E}_u$. As it was already showed in equations (2.11) to (2.15), the total energy per unit length remains constant:

$$\tilde{E}_{tot} = \tilde{E}_{cyl} + \tilde{E}_u = const. \tag{2.85}$$

Using the scaling functions, $\tilde{E}_{cyl}$, $\tilde{E}_u$ and $\tilde{E}_{tot}$ can be written as:

$$\tilde{E}_{cyl} = 2\,\pi\,\frac{\tilde{\rho}_u\,\tilde{l}_c^4}{\tilde{t}_c^2}\,A^2 \int_0^1 I\,\eta\,\mathrm{d}\eta, \tag{2.86}$$

$$\tilde{E}_u = -\pi\,\tilde{p}_u\,\tilde{l}_c^2\,R^2, \tag{2.87}$$

$$\tilde{E}_{tot} = \pi\,\frac{\tilde{\rho}_u\,\tilde{l}_c^4}{\tilde{t}_c^2}\left(2\,A^2 \int_0^1 I\,\eta\,\mathrm{d}\eta - \frac{\tilde{a}^2\,\tilde{t}_c^2}{\tilde{l}_c^2}\,R^2\right), \tag{2.88}$$

with

$$I = \frac{f}{\gamma\,(\gamma-1)} + \frac{\phi^2\,\psi}{2}.$$

Scaling $\tilde{E}_{tot}$, $\tilde{E}_{cyl}$ and $\tilde{E}_u$ with a characteristic energy per unit length $\tilde{E}_c$, it reads:

$$\tilde{E}_{tot} = \tilde{E}_c\,E_{tot}, \tag{2.89}$$

$$\tilde{E}_{cyl} = \tilde{E}_c\,E_{cyl}, \tag{2.90}$$

$$\tilde{E}_u = \tilde{E}_c\,E_u. \tag{2.91}$$

Looking at equations (2.88) and (2.90), three non-dimensional products can be found to determine the characteristic quantities:

$$\Pi_1 = \frac{\tilde{E}_{tot}}{\tilde{E}_c} \tag{2.92}$$

$$\Pi_2 = \frac{\tilde{a}\,\tilde{t}_c}{\tilde{l}_c} \tag{2.93}$$

$$\Pi_3 = \frac{\tilde{\rho}_u\,\tilde{l}_c^4}{\tilde{E}_{tot}\,\tilde{t}_c^2}. \tag{2.94}$$

Setting $\Pi_1 = \Pi_2 = \Pi_3 = 1$ without loss of generality, it reads:

$$\tilde{E}_c = \tilde{E}_{tot}, \tag{2.95}$$

$$\tilde{t}_c = \sqrt{\frac{\tilde{E}_{tot}}{\tilde{\rho}_u}}\, \tilde{a}^{-2}, \tag{2.96}$$

$$\tilde{l}_c = \sqrt{\frac{\tilde{E}_{tot}}{\tilde{\rho}_u}}\, \tilde{a}^{-1}. \tag{2.97}$$

By expanding the total, i.e. initial, energy per unit length E_{tot} and the energy per unit length inside the shock wave cylinder $\tilde{E}_{cyl}$, the energies per unit length conform with:

$$E_{tot} \sim E_{tot,0} + \tau\, E_{tot,1}, \tag{2.98}$$

$$E_{cyl} \sim E_{cyl,0} + \tau\, E_{cyl,1}. \tag{2.99}$$

The energy per unit length E_u scales with R^2, therefore it is of the order $\mathcal{O}(\tau^1)$ and higher, and only the first order will be considered:

$$E_u \sim \tau\, E_{u,1}. \tag{2.100}$$

Therefore, and with the scaling for the energy per unit length, the leading oder terms are:

$$E_{tot,0} = E_{cyl,0} = 1, \tag{2.101}$$

$$E_{tot,0} = 2\,\pi\, A_0^2 \int_0^1 I_0\, \eta\, \mathrm{d}\eta, \tag{2.102}$$

with

$$I_0 = \frac{f_0}{\gamma\,(\gamma-1)} + \frac{\phi_0^2\, \psi_0}{2}.$$

The total energy per unit length remains constant. Thus, the first order terms of E_{cyl} and E_u, i.e. $E_{tot,1}$, must equal zero:

$$E_{tot,1} = E_{cyl,1} + E_{u,1} = 0, \tag{2.103}$$

$$E_{tot,1} = 2\,\pi\, A_0^2 \int_0^1 I_1\,\eta\,\mathrm{d}\eta + 4\,\pi\, A_0\, A_1 \int_0^1 I_0\,\eta\,\mathrm{d}\eta - \frac{\pi}{\gamma(\gamma-1)}\, R_0^2, \tag{2.104}$$

with

$$I_1 = \frac{f_0}{\gamma\,(\gamma-1)} + \phi_0\,\phi_1\,\psi_0 + \frac{\phi_0^2\,\psi_1}{2}.$$

Now A_0 can be expressed using equation (2.102):

$$A_0 = \frac{1}{\sqrt{2\,\pi \int_0^1 I_0\,\eta\,\mathrm{d}\eta}}. \tag{2.105}$$

The value of A_1 has to be know in oder to be able to calculate the correct value of I_1:

$$A_1 = \frac{\pi}{2\,\gamma\,(\gamma-1)}\, A_0\, R_0^2 - \pi\, A_0^3 \int_0^1 I_1\,\eta\,\mathrm{d}\eta. \tag{2.106}$$

As the system of equations is linear in the order $\mathcal{O}(\tau^1)$ terms, two values for A_1 can be set at will and solve those two problems individually. If $A_1 = 0$, the problem is linear and homogeneous, therefore this solution will have the index h, whereas the particular solution will have the index p. $A_1^h = 0$ and $A_1^p = 1$ will be chosen. When the solution to both problems is known, the real value of A_1 can be calculated by superposing the two solutions. The superposed solution must also fulfill the boundary conditions. Therefore, the boundary condition for the homogeneous solution are chosen according to equations (2.80) to (2.82), and for the particular solution the boundary conditions are set zero, $\boldsymbol{x}_1^p(1) = \boldsymbol{0}$. Now A_1 can be calculated using the physical condition, that the velocity must be zero $\tilde{u} = 0$ at $\eta = 0$. That means that there is no source of mass at the center of the shock wave cylinder, i.e. the energy per unit length balance can be satisfied. Equations (2.39) and (2.46) show that this means that $\phi_1(0) = 0$. A_1 will be chosen accordingly:

$$\boldsymbol{x}_1 = \boldsymbol{x}_1^h + A_1\, \boldsymbol{x}_1^p, \tag{2.107}$$

$$\phi_1 = \phi_1^h + A_1\, \phi_1^p, \tag{2.108}$$

$$A_1 = -\frac{\phi_1^h(0)}{\phi_1^p(0)}. \tag{2.109}$$

3 Numerical implementation

The numerical method used to solve the system of ordinary differential equations presented before is the explicit Euler method. The BLAS functions were provided by the GNU Scientific Library. The first system, i.e. equation (2.58), is non-linear, the second one, i.e. equation (2.67), is linear. First, the system of the order $\mathcal{O}(\tau^0)$ problem is solved by starting at $\eta = 1$ and integrating to $\eta = 0$, then A_0 is calculated using equation (2.105). The integral in this equation is solved by the GNU Scientific Library provided function using the QAG adaptive integrating algorithm with the 61 point Gauß-Kronrod rules [6]. Then the solution to the second system is obtained with two chosen values for A_1 ($A_1^h = 0$ and $A_1^p = 1$) as described in section 2.2 starting again at $\eta = 1$ integrating to $\eta = 0$. After getting those solutions, the exact value of A_1 is calculated according to equation (2.109). The two systems of ordinary differential equations are as follows:

$$\boldsymbol{L}_0\, \boldsymbol{x}_0' = \boldsymbol{n}_0, \tag{3.1}$$

$$\boldsymbol{L}_0\, \boldsymbol{x}_1' = \boldsymbol{P}_0\, \boldsymbol{x}_1 + \boldsymbol{m}_0\, A_1. \tag{3.2}$$

These equations are discretised:

$$\boldsymbol{L}_0^{(n)}\, \frac{\boldsymbol{x}_0^{(n)} - \boldsymbol{x}_0^{(n-1)}}{\Delta\eta} = \boldsymbol{n}_0^{(n)}, \tag{3.3}$$

$$\boldsymbol{L}_0^{(n)}\, \frac{\boldsymbol{x}_1^{(n)} - \boldsymbol{x}_1^{(n-1)}}{\Delta\eta} = \boldsymbol{P}_0^{(n)}\, \boldsymbol{x}_1^{(n)} + \boldsymbol{m}_0^{(n)}\, A_1. \tag{3.4}$$

The only unknowns in these equations are $\boldsymbol{x}_0^{(n-1)}$ and $\boldsymbol{x}_1^{(n-1)}$, respectively:

$$\boldsymbol{x}_0^{(n-1)} = \boldsymbol{x}_0^{(n)} - {\boldsymbol{L}_0^{(n)}}^{-1}\, \boldsymbol{n}_0^{(n)}\, \Delta\eta, \tag{3.5}$$

$$\boldsymbol{x}_1^{(n-1)} = \boldsymbol{x}_1^{(n)} - \boldsymbol{L}_0^{(n)^{-1}} \left(\boldsymbol{P}_0^{(n)}\, \boldsymbol{x}_1^{(n)} + \boldsymbol{m}_0^{(n)}\, A_1\right) \Delta\eta. \tag{3.6}$$

Figure 3.1 shows how the systems of differential equations were solved. Noteworthy is that the integrating was done starting at $\eta = 1$ to $\eta = 0$.

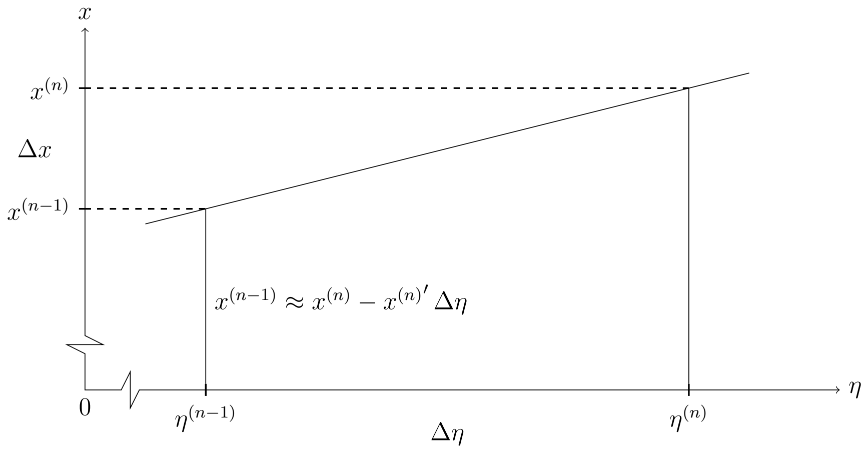
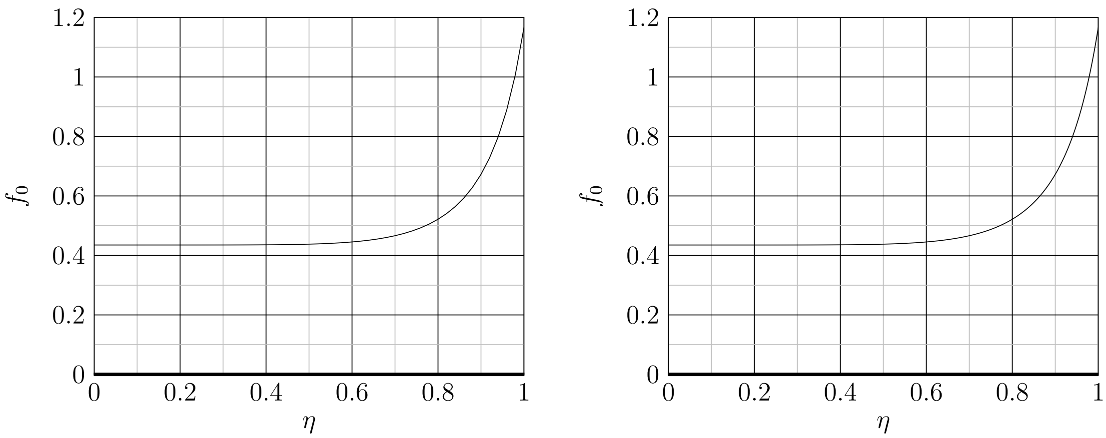


Figure 3.1: Explicit Euler method for integrating.

The number of steps, i.e. the size of $\Delta\eta$, may not have an influence on the results. Therefore, comparing two numerical solutions with different numbers of steps will show if there is a difference between those solutions. If this difference is very small or inexistent, the number of calucation steps is high enough. Figure 3.2 shows that the numerical calculation does not change noticeably choosing $\Delta\eta_1 = 1 \times 10^{-4}$, i.e. $n_1 = 1 \times 10^4$, or $\Delta\eta_2 = 5 \times 10^{-6}$, i.e. $n_2 = 2 \times 10^5$.



(a) $\Delta\eta_1 = 1 \times 10^{-4}$, $n_1 = 1 \times 10^4$ (b) $\Delta\eta_2 = 5 \times 10^{-6}$, $n_2 = 2 \times 10^5$

Figure 3.2: Comparison of f_0 for two different step sizes.

4 Numerical solutions

4.1 Limitations

The approach to solve this problem is only valid when $\tau \ll 1$. The three scaling functions have to fulfill the following physical conditions:

- The pressure p at $\eta = 1$ must be higher than the pressure of the undisturbed surrounding p_u.
- The density ρ at $\eta = 1$ must be higher than the density of the undisturbed surrounding ρ_u
- Looking at figures 4.2 and 4.8, it is possible to find a $\tau_{max,\phi}$ so that the velocity inside the cylinder is zero. This would mean that the shock wave has become too weak to be described correctly by the equations obtained.

Therefore, three conditions are set for τ_{max}. The first condition assumes that the pressure directly behind the shock wave may not be below the pressure of the undisturbed surrounding:

$$\frac{\tilde{p}(\tilde{r},\tilde{t})}{\tilde{p}_u} = \frac{A^2}{R^2} f(\eta(\tilde{r},\tilde{t}), \tau(\tilde{t})), \tag{4.1}$$

$$\tilde{p}(\tilde{R}, \tau_{max,f}) = \tilde{p}_u, \tag{4.2}$$

$$f(1, \tau_{max,f}) = \frac{R^2(\tau_{max,f})}{A^2(\tau_{max,f})}. \tag{4.3}$$

The second condition is that the density immediately behind the shock wave may not fall below the density in the undisturbed surrounding:

$$\frac{\tilde{\rho}(\tilde{r},\tilde{t})}{\tilde{\rho}_u} = \psi(\eta(\tilde{r},\tilde{t}),\tau(\tilde{t})), \tag{4.4}$$

$$\tilde{\rho}(\tilde{R},\tau_{max,\psi}) = \tilde{\rho}_u, \tag{4.5}$$

$$\psi(1,\tau_{max,\psi}) = 1. \tag{4.6}$$

The third condition for τ_{max} says that the velocity inside the shock wave cylinder may not be less than zero. If the velocity is zero, that means that the shock wave has decayed completely, and we cannot fulfil the shock wave equations anymore. Therefore, the assumptions made in chapter 2 are not valid any longer.

4.2 Results

The following parameters were used for the results presented in this section:

- $\tilde{T}_u = 300\,\mathrm{K}$,
- $\tilde{p}_u = 1 \times 10^5\,\mathrm{Pa}$,
- $\gamma = 1.4$,
- $\tilde{E}_i = 5 \times 10^{-2}\,\mathrm{J\,m^{-1}}$,
- $\tilde{R}_u = 8.314\,\mathrm{J\,K^{-1}\,mol^{-1}}$,
- $\tilde{M} = 2.897\,\mathrm{kg\,mol^{-1}}$,
- $n = 2 \times 10^6$.

Using these parameters for the numerical calculations, the calculated parameters conform with:

- $A_0 = 0.504$,
- $A_1 = 1.983$,
- $l_c = 5.976 \times 10^{-4}\,\mathrm{m}$,
- $R_0 = 1.004$,
- $R_1 = 0.988$,
- $t_c = 1.721 \times 10^{-6}\,\mathrm{s}$,
- $\tau_{max,f} = 0.853$,
- $\tau_{max,\phi} = 0.182$,
- $\tau_{max,\psi} = 0.042$.

Table 4.1: Results of numerical calculation.

η	f_0	ϕ_0	ψ_0	f_1	ϕ_1	ψ_1	f_1^h	ϕ_1^h	ψ_1^h	f_1^p	ϕ_1^p	ψ_1^p
1.00	+1.167	+0.833	+6.000	−0.661	−3.307	−119.0	−0.661	−3.307	−119.0	+0.000	+0.000	+0.000
0.95	+0.842	+0.761	+3.182	+2.396	−3.109	−3.748	−0.684	−4.427	−16.28	+1.553	+0.665	+6.323
0.90	+0.673	+0.698	+1.884	+1.756	−3.033	+12.49	−3.757	−6.448	+2.203	+2.780	+1.722	+5.189
0.85	+0.578	+0.643	+1.193	+0.858	−2.987	+12.15	−6.986	−9.314	+6.140	+3.955	+3.190	+3.035
0.80	+0.522	+0.593	+0.786	+0.185	−2.923	+9.380	−9.872	−12.90	+7.659	+5.071	+5.033	+0.868
0.75	+0.488	+0.549	+0.528	−0.265	−2.825	+6.779	−12.35	−17.11	+8.813	+6.095	+7.203	−1.025
0.70	+0.466	+0.508	+0.356	−0.553	−2.695	+4.748	−14.45	−21.87	+9.710	+7.007	+9.671	−2.502
0.65	+0.453	+0.468	+0.238	−0.735	−2.540	+3.243	−16.19	−27.20	+10.17	+7.794	+12.43	−3.494
0.60	+0.445	+0.431	+0.156	−0.846	−2.367	+2.157	−17.60	−33.17	+10.09	+8.452	+15.53	−4.000
0.55	+0.441	+0.394	+0.100	−0.912	−2.182	+1.389	−18.72	−39.90	+9.466	+8.981	+19.02	−4.073
0.50	+0.438	+0.358	+0.062	−0.950	−1.991	+0.860	−19.57	−47.62	+8.395	+9.390	+23.00	−3.800
0.45	+0.436	+0.322	+0.036	−0.971	−1.795	+0.507	−20.19	−56.63	+7.030	+9.691	+27.65	−3.289
0.40	+0.436	+0.286	+0.020	−0.982	−1.597	+0.281	−20.61	−67.44	+5.539	+9.900	+33.20	−2.651
0.35	+0.435	+0.250	+0.010	−0.987	−1.398	+0.144	−20.89	−80.83	+4.078	+10.03	+40.05	−1.984
0.30	+0.435	+0.214	+0.005	−0.989	−1.199	+0.067	−21.05	−98.08	+2.773	+10.11	+48.85	−1.364
0.25	+0.435	+0.179	+0.002	−0.990	−0.999	+0.027	−21.14	−121.5	+1.706	+10.16	+60.77	−0.847
0.20	+0.435	+0.143	+0.001	−0.990	−0.799	+0.009	−21.18	−155.8	+0.916	+10.18	+78.17	−0.458
0.15	+0.435	+0.107	+0.000	−0.990	−0.599	+0.002	−21.20	−211.8	+0.400	+10.19	+106.5	−0.201
0.10	+0.435	+0.071	+0.000	−0.990	−0.400	+0.000	−21.20	−322.1	+0.122	+10.19	+162.2	−0.061
0.05	+0.435	+0.036	+0.000	−0.990	−0.200	+0.000	−21.20	−649.3	+0.015	+10.19	+327.3	−0.008
0.00	+0.435	+0.000	+0.000	−0.990	+0.000	+0.000	−21.20	$-\infty$	+0.000	+10.19	$+\infty$	+0.000

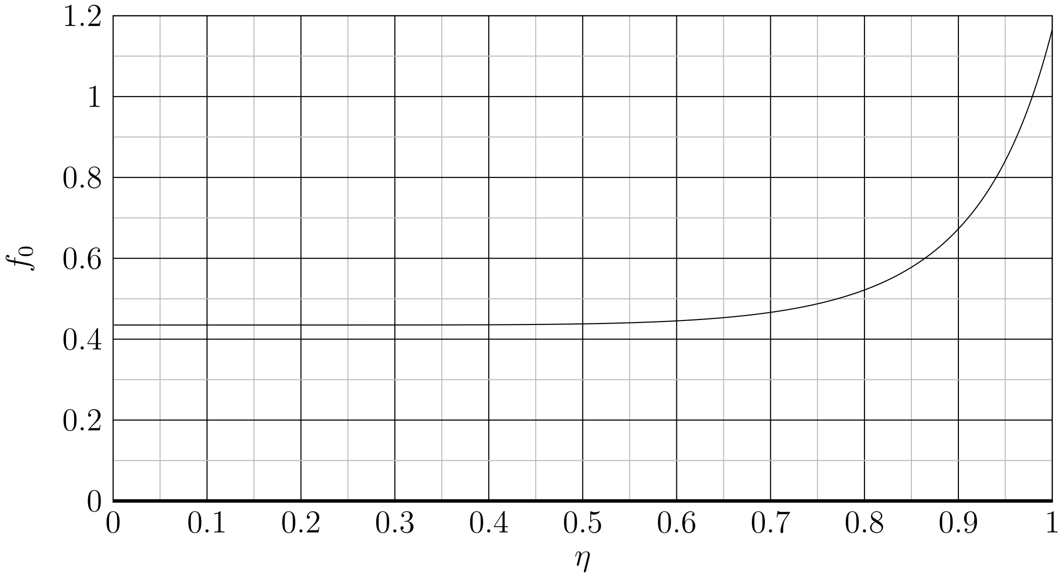


Figure 4.1: Solution of f_0.

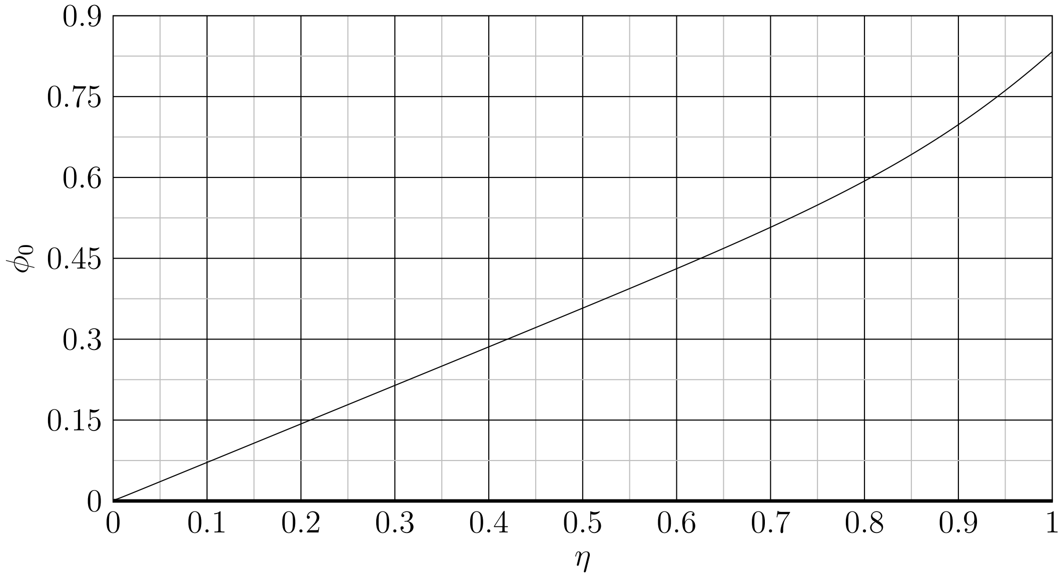


Figure 4.2: Solution of ϕ_0.

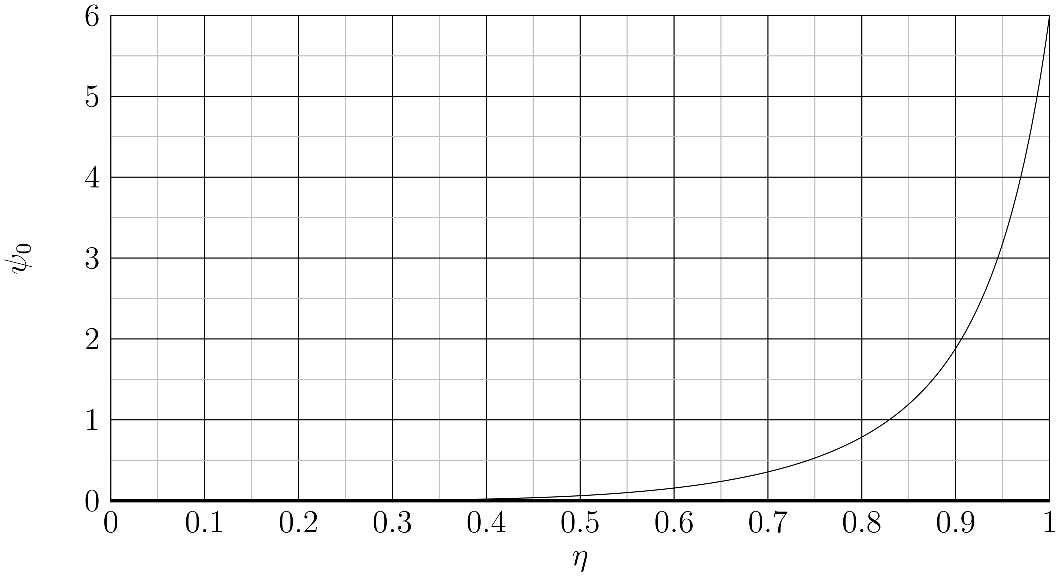


Figure 4.3: Solution of ψ_0.

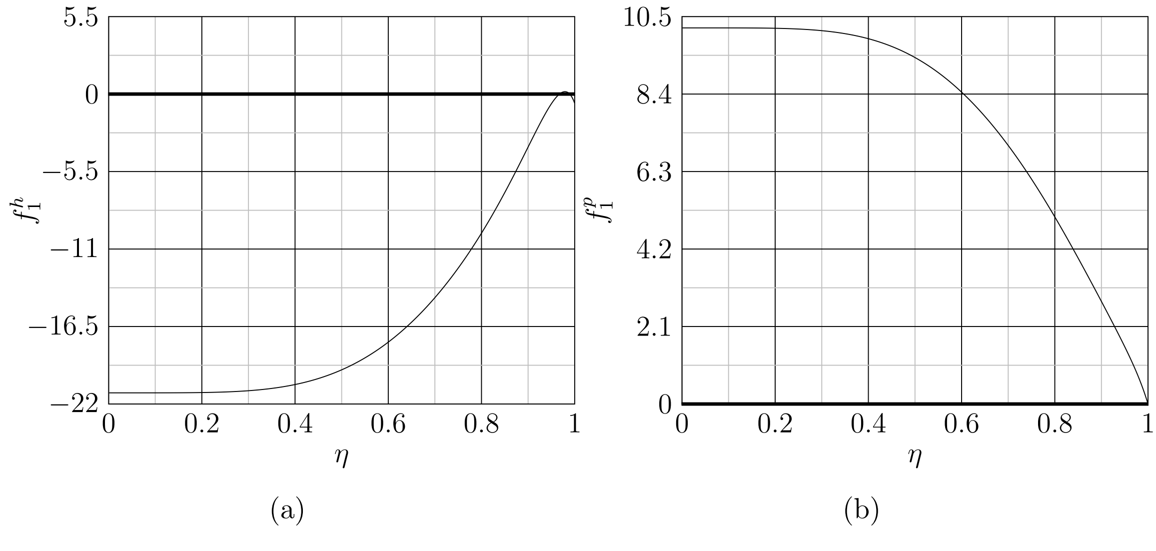
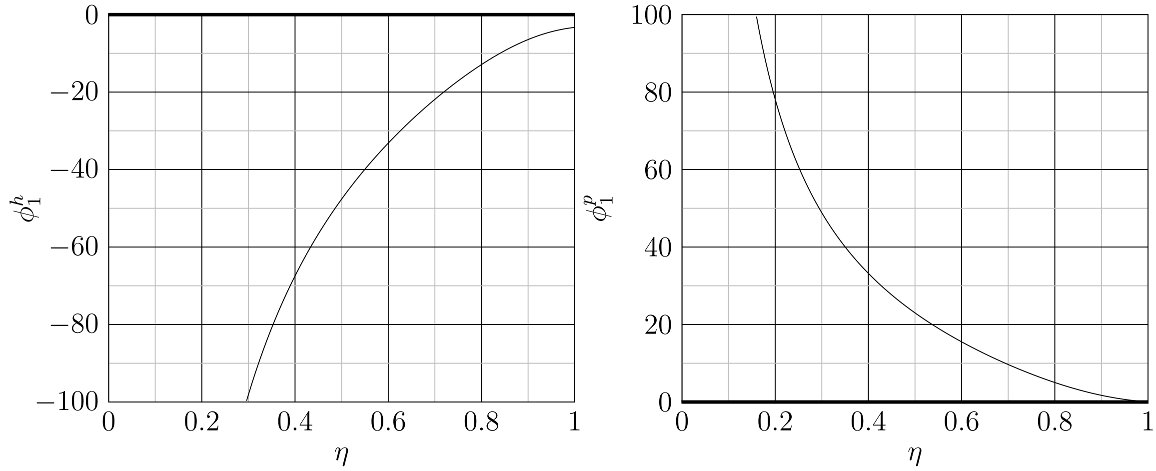
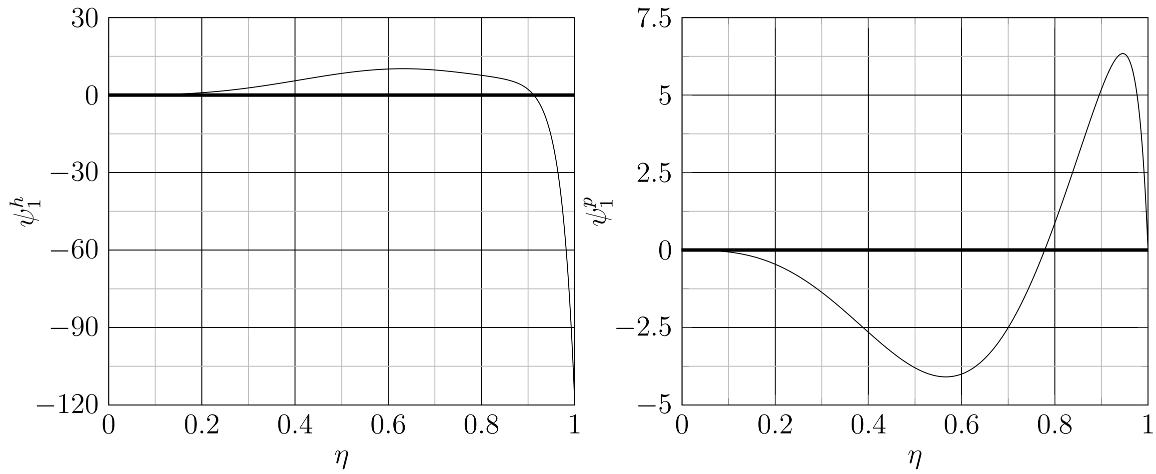


Figure 4.4: Homogeneous (a) and particular (b) solution of f_1.



(a) (b)

Figure 4.5: Homogeneous (a) and particular (b) solution of ϕ_1.



(a) (b)

Figure 4.6: Homogeneous (a) and particular (b) solution of ψ_1.

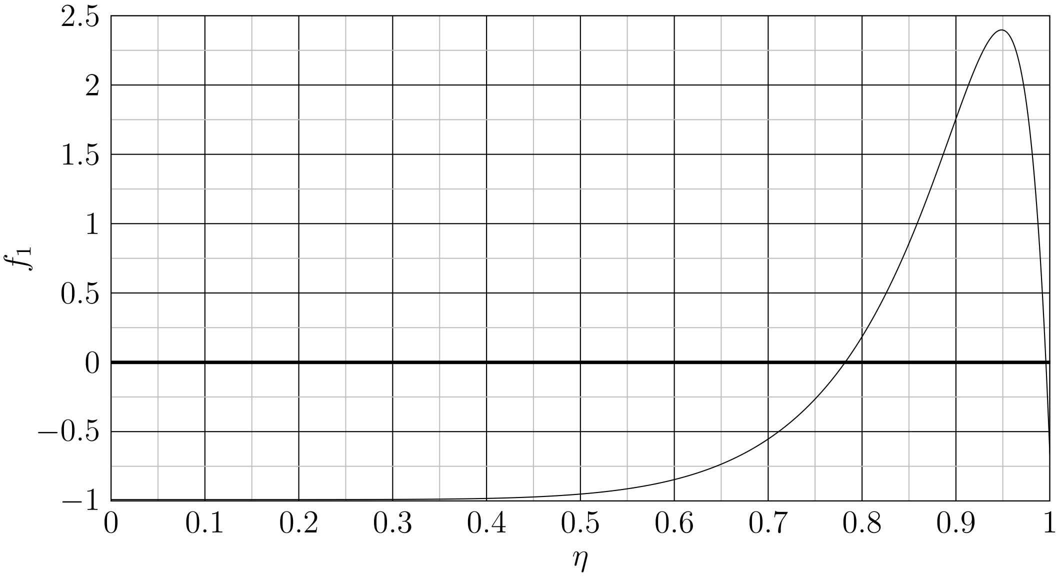


Figure 4.7: Solution of f_1.

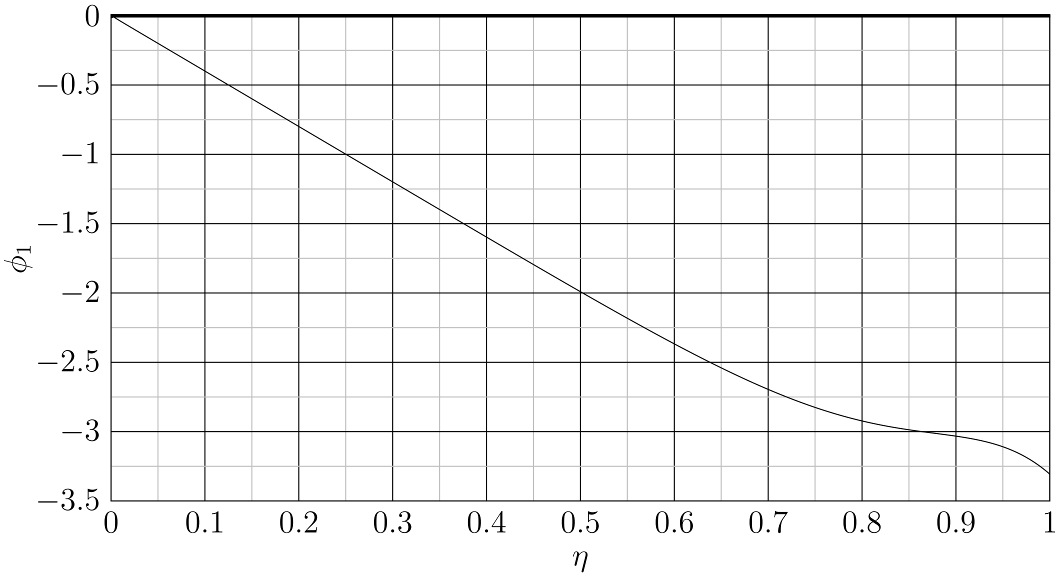


Figure 4.8: Solution of ϕ_1.

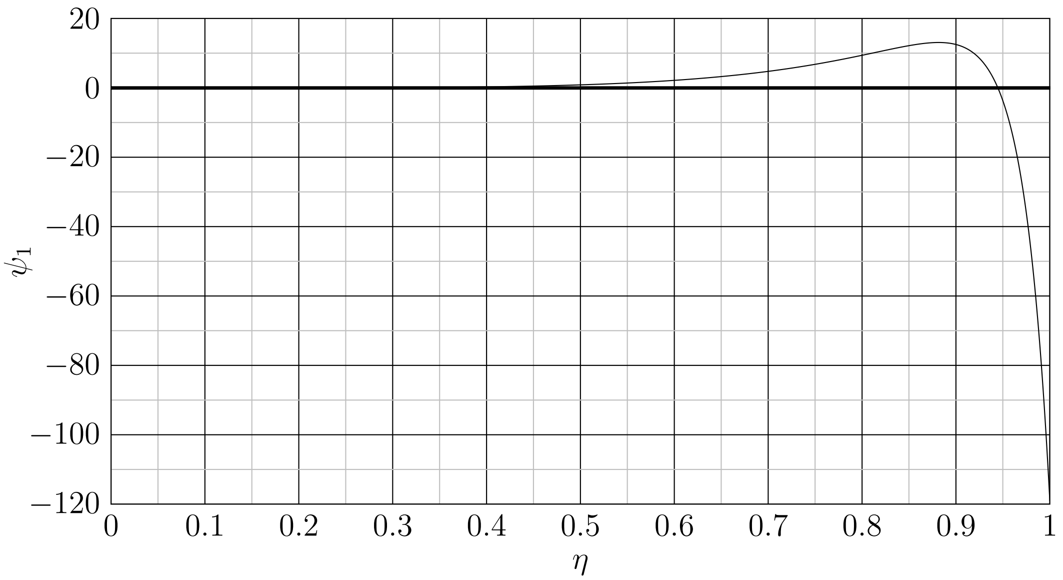


Figure 4.9: Solution of ψ_1.

5 Conclusions

The simplifications made in chapter 2.2.1 were necessary because it was not possible to solve the system of partial differential equations numerically directly. The problems occuring were primarily the following: when η approaches zero, the results for ϕ were physically impossible.

Although there are two different equations to calculate the value of A_1, implementing those two equations numerically will give the same result. This shows that the energy per unit length balance and the physical assumption that the velocity at $\eta = 0$, respectively, are valid and interchangeable

The rather small values of τ_{max} indicate, that terms of higher order than $\mathcal{O}(\tau^1)$ become significant quite soon. Therefore,they have to be considered in calcualtions after a rather short period of time.

However, the solution procedure outlined before may be extended and used in a simple numerical scheme integrating with respect to time: By setting a solution for a very small value of $\Delta\tau \ll \tau_{max}$ as the new order $\mathcal{O}(\tau^0)$ solution, τ can be increased step by step:

$$\boldsymbol{x}^{(0)} = \boldsymbol{x}_0 + \Delta\tau\,\boldsymbol{x}_1 \tag{5.1}$$

$$\boldsymbol{x}_0^{(m)} := \boldsymbol{x}^{(m-1)} \tag{5.2}$$

$$\boldsymbol{x}^{(m)} = \boldsymbol{x}_0^{(m)} + \Delta\tau\,\boldsymbol{x}_1^{(m)}. \tag{5.3}$$

This simple extension allows to calculate $\boldsymbol{x}$ for $\tau > \tau_{max}$ without having to calculate higher order terms. The vadility of this extension has to be checked seperately and will not be discussed in this thesis.

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