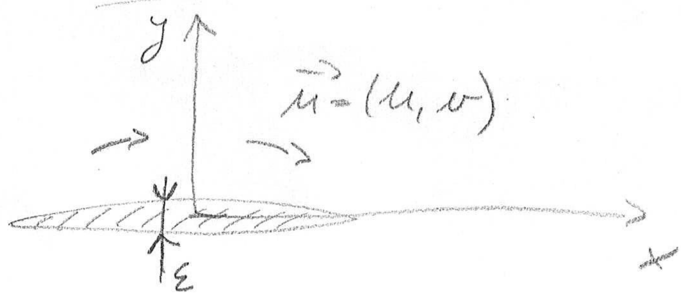
Solution of the wave equation with the method of characteristics

$\vec{u}_0 = (u_0, v_0)$



Potential flow assumption:

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}$$

Far field:

$u_0 = 1$

$v_0 = 0$

$\phi_0 = x$

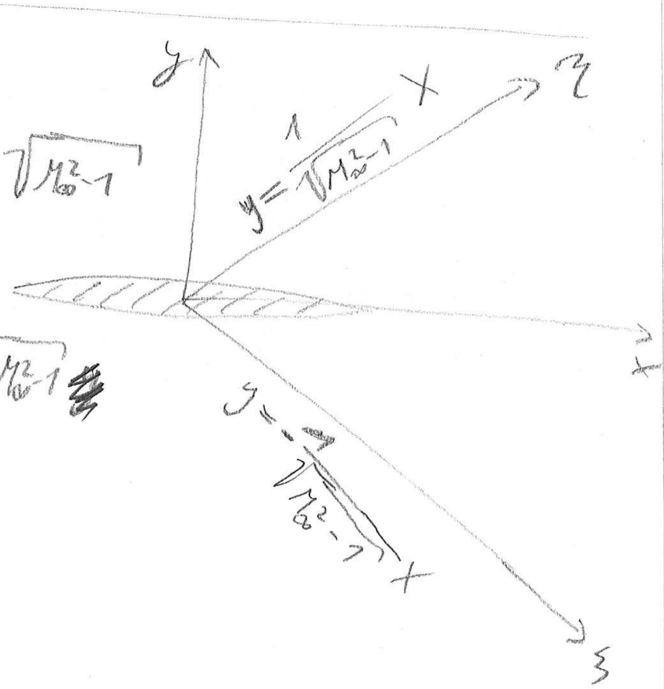
Asymptotic expansion for $\varepsilon \ll 1$:

$$\phi = \phi_0 + \varepsilon \phi_1 = x + \varepsilon \phi_1$$

Coordinate transformation:

$$\xi = x - \sqrt{M_\infty^2 - 1}\, y \qquad \frac{\partial \xi}{\partial x} = 1 \qquad \frac{\partial \xi}{\partial y} = -\sqrt{M_\infty^2 - 1}$$

$$\eta = x + \sqrt{M_\infty^2 - 1}\, y \qquad \frac{\partial \eta}{\partial x} = 1 \qquad \frac{\partial \eta}{\partial y} = \sqrt{M_\infty^2 - 1}$$



$$\phi_1(x, y) = \tilde{\phi}_1(\xi(x, y), \eta(x, y))$$

$$\frac{\partial \phi_1}{\partial x} = \frac{\partial \tilde{\phi}}{\partial \xi} \underbrace{\frac{\partial \xi}{\partial x}}_{1} + \frac{\partial \tilde{\phi}}{\partial \eta} \underbrace{\frac{\partial \eta}{\partial x}}_{1} = \frac{\partial \tilde{\phi}}{\partial \xi} + \frac{\partial \tilde{\phi}}{\partial \eta}$$

$$\frac{\partial \phi_1}{\partial y} = \frac{\partial \tilde{\phi}}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \tilde{\phi}}{\partial \eta} \frac{\partial \eta}{\partial y} = -\sqrt{M_\infty^2 - 1}\, \frac{\partial \tilde{\phi}}{\partial \xi} + \sqrt{M_\infty^2 - 1}\, \frac{\partial \tilde{\phi}}{\partial \eta}$$

P.O

Supersonic flow over a slender airfoil:

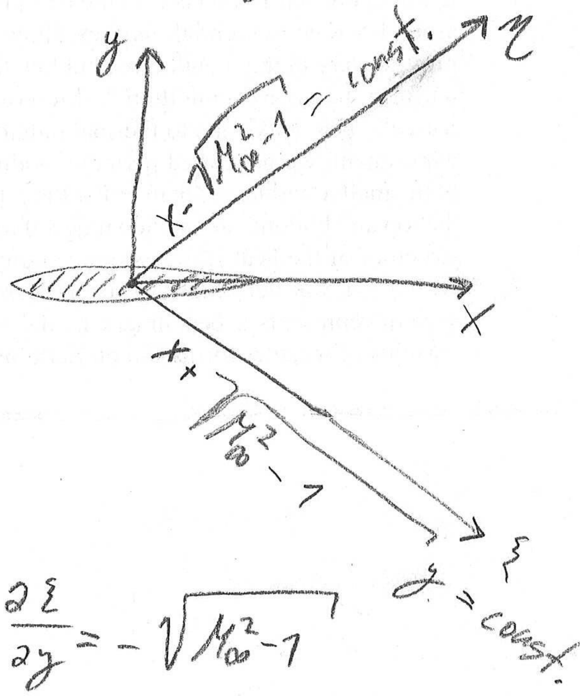
D'Alembert solution of the wave equation

$$(1-M_\infty^2)\,\phi_{1,xx} + \phi_{1,yy} = 0$$

Characteristics: $\dfrac{dy_{1,2}}{dx} = \pm \dfrac{1}{\sqrt{M_\infty^2-1}}$ or $\dfrac{dx_{1,2}}{dy} = \pm\sqrt{M_\infty^2-1}$

$$\int \frac{dx}{dy}\,dy \Rightarrow x_1 = C_1 + \sqrt{M_\infty^2-1}\; y$$

$$x_2 = C_2 - \sqrt{M_\infty^2-1}\; y$$



Coordinate transformation:

$$\xi = x - \sqrt{M_\infty^2-1}\, y \;,\quad \frac{\partial \xi}{\partial x} = 1 \;,\quad \frac{\partial \xi}{\partial y} = -\sqrt{M_\infty^2-1}$$

$$\eta = x + \sqrt{M_\infty^2-1}\, y \;,\quad \frac{\partial \eta}{\partial x} = 1 \;,\quad \frac{\partial \eta}{\partial y} = \sqrt{M_\infty^2-1}$$

Search for solution: $\phi_1(x,y) = \tilde{\phi}\big(\xi(x,y), \eta(x,y)\big)$

$$\frac{\partial \phi_1}{\partial x} = \frac{\partial \tilde\phi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \tilde\phi}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial \tilde\phi}{\partial \xi} + \frac{\partial \tilde\phi}{\partial \eta}$$

$$\frac{\partial \phi_1}{\partial y} = \frac{\partial \tilde\phi}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \tilde\phi}{\partial \eta}\frac{\partial \eta}{\partial y} = -\sqrt{M_\infty^2-1}\,\frac{\partial \tilde\phi}{\partial \xi} + \sqrt{M_\infty^2-1}\,\frac{\partial \tilde\phi}{\partial \eta}$$

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$$\cancel{\frac{\partial^2 \tilde{\theta}}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial \tilde{\theta}}{\partial \xi} + \frac{\partial \tilde{\theta}}{\partial \eta}\right) = \frac{\partial^2 \tilde{\theta}}{\partial \xi^2} + 2\frac{\partial^2 \tilde{\theta}}{\partial \xi \partial \eta} + \frac{\partial^2 \tilde{\theta}}{\partial \eta^2}}$$

$$\frac{\partial^2 \tilde{\theta}}{\partial x^2} = \overbrace{\left(\underbrace{\frac{\partial \xi}{\partial x}}_{1}\frac{\partial}{\partial \xi} + \underbrace{\frac{\partial \eta}{\partial x}}_{1}\frac{\partial}{\partial \eta}\right)}^{\partial/\partial x}\frac{\partial \tilde{\theta}}{\partial x}$$

$$= \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial \tilde{\theta}}{\partial \xi} + \frac{\partial \tilde{\theta}}{\partial \eta}\right) = \frac{\partial^2 \tilde{\theta}}{\partial \xi^2} + 2\frac{\partial^2 \tilde{\theta}}{\partial \xi \partial \eta} + \frac{\partial^2 \tilde{\theta}}{\partial \eta^2}$$

$$\frac{\partial^2 \tilde{\theta}}{\partial y^2} = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}\right)\frac{\partial \tilde{\theta}}{\partial y}$$

$$= \left(-\sqrt{M_\infty^2 - 1}\,\frac{\partial}{\partial \xi} + \sqrt{M_\infty^2 - 1}\,\frac{\partial}{\partial \eta}\right)\left(-\sqrt{M_\infty^2 - 1}\,\frac{\partial \tilde{\theta}}{\partial \xi} + \sqrt{M_\infty^2 - 1}\,\frac{\partial \tilde{\theta}}{\partial \eta}\right)$$

$$= \left(M_\infty^2 - 1\right)\frac{\partial^2 \tilde{\theta}}{\partial \xi^2} - 2\left(M_\infty^2 - 1\right)\frac{\partial^2 \tilde{\theta}}{\partial \xi \partial \eta} + \left(M_\infty^2 - 1\right)\frac{\partial^2 \tilde{\theta}}{\partial \eta^2}$$

$$-\left(M_\infty^2 - 1\right)\frac{\partial^2 \hat{\theta}_1}{\partial x^2} + \frac{\partial^2 \hat{\theta}_1}{\partial y^2} = 0$$

$$-\left(M_\infty^2 - 1\right)\left(\frac{\partial^2 \tilde{\theta}}{\partial \xi^2} + 2\frac{\partial^2 \tilde{\theta}}{\partial \xi \partial \eta} + \frac{\partial^2 \tilde{\theta}}{\partial \eta^2}\right) + \left(M_\infty^2 - 1\right)\left(\frac{\partial^2 \tilde{\theta}}{\partial \xi^2} - 2\frac{\partial^2 \tilde{\theta}}{\partial \xi \partial \eta} + \frac{\partial^2 \tilde{\theta}}{\partial \eta^2}\right) = 0$$

$$-4\left(M_\infty^2 - 1\right)\frac{\partial^2 \tilde{\theta}}{\partial \xi \partial \eta} = 0$$

$$\frac{\partial^2 \tilde{\theta}}{\partial \xi \partial \eta} = 0$$

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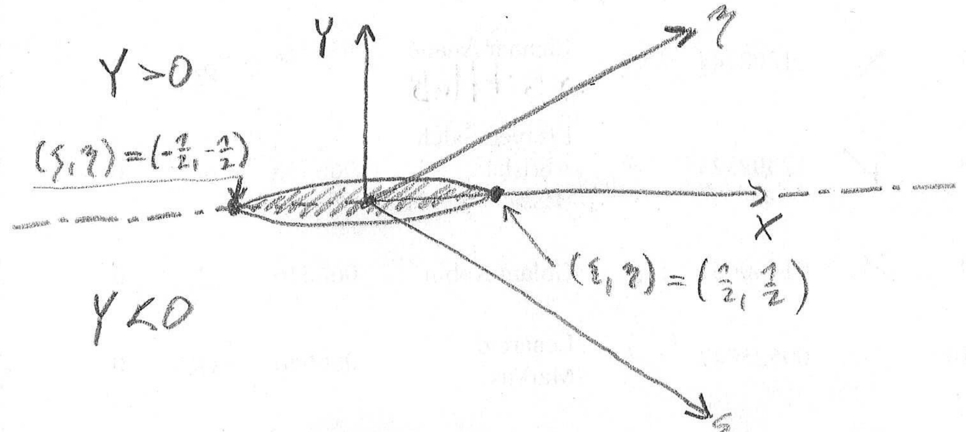
$$\frac{\partial^2 \tilde{\Phi}}{\partial\xi\partial\eta} = 0 \implies \text{solution: } \tilde{\Phi} = A(\xi) + B(\eta)$$

$$\Phi_1(X,Y) = A\left(X - \sqrt{M_\infty^2 - 1}\,Y\right) + B\left(X + \sqrt{M_\infty^2 - 1}\,Y\right)$$

A and B are determined by the boundary conditions:

- Inflow condition: $\Phi_1 \to 0$ as $X \to 0$ (4.24)



For $Y>0$: $\tilde{\Phi} \to 0$ as $\xi \to -\infty$

(note that $X \to -\infty$ as $\xi \to -\infty$)

$\implies$ then, $B(\eta) = 0$ for $Y > 0$. $A \to 0$ as $\xi \to -\infty$.

[Note: in general, $B(\eta) = \text{const.}$ and $A(-\infty) = -B$. Setting $B=0$ is convenient]

For $Y<0$: $\tilde{\Phi} \to 0$ as $\eta \to -\infty$

$\implies A(\xi) = 0$ for $Y < 0$.

$B(\eta) \to 0$ as $\eta \to -\infty$.

Thus,
$$\frac{\partial \Phi_1}{\partial Y} = \begin{cases} \dfrac{\partial \xi}{\partial Y}\dfrac{dA}{d\xi} & \text{if } Y>0 \\[2ex] \dfrac{\partial \eta}{\partial Y}\dfrac{dB}{d\eta} & \text{if } Y<0 \end{cases}$$

• Kinematic condition: (4.30) and (4.32):

For $-\frac{1}{2} \le X \le \frac{1}{2}$:

$$\frac{\partial \Phi_1}{\partial Y} = \begin{cases} -\sqrt{M_\infty^2 - 1}\, A' = H' & \text{if } Y = 0^+ \\ \sqrt{M_\infty^2 - 1}\, B' = -H' & \text{if } Y = 0^- \end{cases}$$

• Symmetry condition (4.26):

For $X < -\frac{1}{2}$ or $X > \frac{1}{2}$:

$$\frac{\partial \Phi_1}{\partial Y} = 0 \quad \text{at } Y = 0$$

$$\frac{\partial \Phi_1}{\partial Y} = \begin{cases} -\sqrt{M_\infty^2 - 1}\, A' = 0 & \text{at } Y = 0^+ \\ \sqrt{M_\infty^2 - 1}\, B' = 0 & \text{at } Y = 0^- \end{cases}$$

! Note: At $Y = 0$, $\xi = \eta = X$.

Integrate the symmetry conditions:

For $Y=0^+$ and $\xi < -\frac{1}{2}$: $A' = 0$

$$A(\xi) = \text{const.} = a$$

For $Y=0^-$ and $\eta < -\frac{1}{2}$: $B' = 0$

$$B(\eta) = \text{const.} = b$$

Then, match the constants a, b to the inflow B.C.:

$A \to 0$ as $\xi \to -\infty \Rightarrow \underline{a = 0}$

$B \to 0$ as $\eta \to -\infty \Rightarrow \underline{b = 0}$

For $Y=0^+$ and $\xi > \frac{1}{2}$: $A' = 0$

$$A(\xi) = \text{const.} = c$$

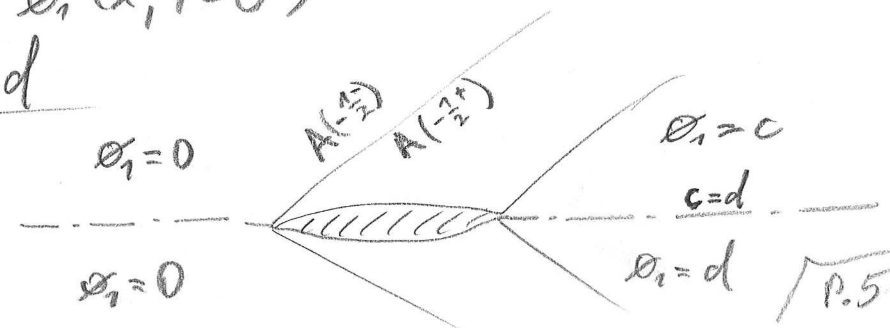
For $Y=0^-$ and $\eta > \frac{1}{2}$: $B' = 0$

$$B(\eta) = d$$

Furthermore, we require a continuous Φ_1 for $X > \frac{1}{2}$ and $Y=0$:

$$\Phi_1(X, Y=0^+) = \Phi_1(X, Y=0^-)$$

$$\underline{c = d}$$



Integrate the kinematic condition:

For $Y=0^+$ and $-\frac{1}{2} \leq \xi \leq \frac{1}{2}$:

$$-\sqrt{M_\infty^2 - 1}\, A' = H'$$

$$\underline{A = -(M_\infty^2 - 1)^{-1/2} H + e}$$

For $Y=0^-$ and $-\frac{1}{2} \leq \xi \leq \frac{1}{2}$:

$$B' = -(M_\infty^2 - 1)^{-1/2} H'$$

$$\underline{B = -(M_\infty^2 - 1)^{-1/2} H + f}$$

Determine constants e, f from continuity of $\tilde{\sigma}$:

(Assume $H(-\frac{1}{2}) = H(\frac{1}{2}) = 0$)

$$A(-\tfrac{1}{2}^+) = A(-\tfrac{1}{2}^-) \qquad B(-\tfrac{1}{2}^+) = B(-\tfrac{1}{2}^-)$$

$$\underline{e = 0} \qquad \underline{f = 0}$$

Finally, determine $c = d$:

$$A(\xi \to \tfrac{1}{2}^+) = A(\xi \to \tfrac{1}{2}^-)$$

$$\underline{\underline{c = 0}}$$

Thus, we receive the following solution for $M_\infty > 1$:

$$\phi_1(x,y) = \tilde{\phi}(\xi,\eta) = \begin{cases} 0 & \text{for } Y \geq 0 \text{ and } \xi \leq -\frac{1}{2} \\ 0 & \text{for } Y \leq 0 \text{ and } \eta \leq -\frac{1}{2} \\ 0 & \text{for } Y \geq 0 \text{ and } \xi \geq \frac{1}{2} \\ 0 & \text{for } Y \leq 0 \text{ and } \eta \geq \frac{1}{2} \\ -(M_\infty^2-1)^{-1/2} H(\xi) & \text{for } Y \geq 0 \text{ and } -\frac{1}{2} \leq \xi \leq \frac{1}{2} \\ -(M_\infty^2-1)^{-1/2} H(\eta) & \text{for } Y \leq 0 \text{ and } -\frac{1}{2} \leq \eta \leq \frac{1}{2} \end{cases}$$

Recall: $u = \frac{\partial \phi}{\partial X}$, $V = \frac{\partial \phi}{\partial Y}$

$\phi = \phi_0 + \varepsilon \phi_1$, where $\varepsilon \ll 1$ is the dimensionless thickness of the profile

$\phi_0 = X$

$U = 1 + \varepsilon \frac{\partial \phi_1}{\partial X}$; $\frac{\partial \phi_1}{\partial X} = \frac{\partial \tilde{\phi}}{\partial \xi}\frac{\partial \xi}{\partial X} + \frac{\partial \tilde{\phi}}{\partial \eta}\frac{\partial \eta}{\partial X}$

$V = \varepsilon \frac{\partial \phi_1}{\partial Y}$; $\frac{\partial \phi_1}{\partial Y} = \frac{\partial \tilde{\phi}}{\partial \xi}\frac{\partial \xi}{\partial Y} + \frac{\partial \tilde{\phi}}{\partial \eta}\frac{\partial \eta}{\partial Y}$