Computation of the laminar boundary layer: Walz method

- Assumption: Velocity profile described locally by Falkner-Skan equation:

$$\boxed{f''' + ff'' + \beta(1-f'^2) = 0} \qquad (7.15)$$

BCs:

$$f=0,\ f'=0 \ \text{at} \ \eta=0 \qquad (7.16)$$
$$f'=1 \ \text{at} \ \eta\to\infty$$

- Variation of $\beta(x)$ determined by the momentum-integral equation:

$$\boxed{\frac{d}{dx}(\delta_2 U^2) + \delta_1 U \frac{dU}{dx} = \frac{\tau_w}{\rho}} \qquad (8.1)\ (7.100)$$

(dimensional)

where $U^2(x)\,\underset{\substack{\uparrow\\ \text{momentum}\\ \text{thickness}}}{\delta_2}(\beta) = \int_0^\infty u(U-u)\,dy$ (7.98) ; $\underset{\substack{\uparrow\\ \text{displacement}\\ \text{thickness}}}{\delta_1}(\beta)\,U(x) = \int_0^\infty (U-u)\,dy$ (7.99)

and $$\frac{\tau_w(\beta)}{\rho} = \nu \left.\frac{\partial u}{\partial y}\right|_w . \qquad (8.5)$$

• Define the similarity coordinate η as follows:

$$\eta = \frac{y}{\delta_N(x)} \;, \quad \delta_N = \sqrt{\frac{2\nu x}{U(m+1)}} \;, \quad m = \frac{\beta}{2-\beta} \qquad (8.2)\ (7.19)$$

estimate of the b.l. thickness

• Then, refer the displacement (δ_1) and momentum (δ_2) thickness to δ_N:

$$\delta_1 = \beta_1 \, \delta_N \;, \quad \delta_2 = \beta_2 \, \delta_N \qquad (8.3,\ 8.4)$$

$$\beta_1 = \frac{\delta_1}{\delta_N} = \int_0^\infty (1-f')\, d\eta = \lim_{\eta\to\infty} (\eta - f) \qquad (8.3)$$

$$\beta_2 = \frac{\delta_2}{\delta_N} = \int_0^\infty (1-f')f'\, d\eta = \frac{f''_w - \beta\beta_1}{1+\beta} \qquad (8.4)$$

• The wall shear stress τ_w is related to the dimensionless stream function f as follows:

$$\frac{\tau_w}{\rho} = \nu \left(\frac{\partial u}{\partial y}\right)_w = \frac{\nu U}{\delta_N} f''_w \qquad (8.5)$$

$\Rightarrow$ Substituting (8.3)-(8.5) into the momentum-integral Eq. (8.1), one obtains (8.7)

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Computation of laminar boundary layer using Hartree profiles

- Follows Chapter 8.1 of Schlichting & Gersten (2017)
- Assume that the velocity profile corresponds locally to the Hartree profile, described by the Falkner-Skan eq. (single parameter β)
- The variation of the boundary-layer thicknesses δ_1 and δ_2 is described by the momentum-integral equation.

The similarity variables are defined as follows:

$$\eta = \frac{y}{\delta_N(x)}, \quad \delta_N = \sqrt{\frac{2\nu x}{U(m+1)}}, \quad m = \frac{\beta}{2-\beta} \qquad (8.2)$$

The displacement thickness δ_1 and momentum thickness δ_2 are related to the scaling δ_N as follows:

$$\delta_1 = \beta_1 \delta_N, \quad \beta_1 = \lim_{\eta \to \infty} (\eta - f) \qquad (8.3)$$

$$\delta_2 = \beta_2 \delta_N, \quad \beta_2 = \frac{f''_w - \beta \beta_1}{1+\beta} \qquad (8.4)$$

(8.5)

The wall shear stress is given by: $\frac{\tau_w}{\rho} = \nu \frac{\partial u}{\partial y}\Big|_w = \frac{\nu U}{\delta_N} f''_w$

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- Remember that all the quantities B_1, B_2, f''_w depend on β, because the profile $f(\eta)$ depends on β through the Falkner-Skan equation.

- Substituting (8.3)-(8.5) into (8.2) one obtains (8.7) in Schlichting & Gersten, p. 198:

$$\frac{d(B_2 \delta_N)}{dx} + \left(2 + \frac{B_1}{B_2}\right) \frac{B_2 \delta_N}{U} \frac{dU}{dx} = \frac{\nu}{U \delta_N} f''_w \qquad (8.7)$$

- Another relationship between the boundary-layer parameters and the outer pressure gradient is provided by the "compatibility condition at the wall."

Compatibility condition at the wall:

If we express the boundary-layer equation for x-momentum at $y=0$, using the boundary conditions, we obtain the "compatibility condition":

$$\mu \left(\frac{\partial^2 u}{\partial y^2}\right)_w = \frac{dp}{dx} = -U \frac{dU}{dx} \qquad (7.2),\ (8.8)$$

$$\left(\frac{\partial^3 u}{\partial y^3}\right)_w = 0 \qquad (7.3)$$

See also chapter 7.1 of S&G, p. 166.

For the Hartree profiles, the compatibility cond. provides additional relationship between param.

For the Hartree profiles, the compatibility condition provides the following relationship:

$$f_w''' = -\beta = -\frac{\delta_N^2}{\nu}\frac{dU}{dx} \qquad (8.9)$$

see Falkner-Skan

A. Waltz (1966):

Instead of finding an equation for $\beta(x)$, Waltz (1966) defined the following "shape factor":

$$\Gamma = -\frac{\delta_2^2}{U}\left(\frac{\partial^2 u}{\partial y^2}\right)_w = \beta \beta_2^2 \qquad (8.13), (8.14)$$

and the following auxiliary variable:

$$Z = \frac{\delta_2^2}{\nu} U \qquad (8.12)$$

Using the momentum-integral equation and the compatibility condition, one obtains the following system of ODEs for $\Gamma(x)$ and $Z(x)$:

compatibility cond.: $$\Gamma = \frac{Z}{U}\frac{dU}{dx} \qquad (8.16)$$

momentum-integral: $$\frac{dZ}{dx} = \overbrace{\underbrace{2\beta_2 f_w''}_{F_1} - \left(3 + 2\underbrace{\frac{\beta_1}{\beta_2}}_{H_{12}}\right)\Gamma}^{F_2} \qquad (8.15)$$

The advantage of introducing the variables Z and Γ is that the RHS of Eq. (8.15) is almost a linear function of Γ. This can be shown by solving the Falkner-Skan eqs. for a range of β and plotting F_2 versus Γ (see Fig. 8.1 on p. 199 of S&G).

Thus, we can replace the RHS of (8.15) with the linear approximation:

$$F_2 = a - b\Gamma \quad \text{where } a = 0.441 \text{ and } b = \begin{cases} 4.165 & \text{for } \Gamma > 0 \\ 4.579 & \text{for } \Gamma < 0 \end{cases} \quad (8.20), (8.24)$$

momentum-integral eq:
(8.15)
$$\overset{(8.20)}{\Longrightarrow} \quad \frac{dZ}{dx} \approx a - b\Gamma \approx a - b\frac{Z}{U}\frac{dU}{dx} \quad (8.21)$$

The variation $Z(x)$ is then obtained by integrating Eq. (8.21), starting from some known initial condition.

At the stagnation point, which is the origin of the b.L., the initial condition

$$Z = 0$$

must be satisfied. We select the stagnation point as the origin of the x-coordinate. ($x=0$)

Computation of laminar boundary layer

The variation of the shape factor for Hartree profiles can be approximated with the following ODE:

Exponential growth (blow-up) for $Z\frac{dU}{dx}<0$!

$$\frac{dZ}{dx} = a - b(\Gamma)\frac{dU}{dx}\frac{Z}{U} \qquad (8.21)$$

undefined at $x=0$!

$$\Gamma = \frac{Z}{U}\frac{dU}{dx} \qquad (8.13)$$

Discretization

IC: $Z(0)=0$

The simplest and cheapest discretization scheme for the initial-value problem is explicit Euler:

$$\frac{Z_{i+1}-Z_i}{x_{i+1}-x_i} = a - b(\Gamma_i)\frac{U_{i+1}-U_i}{x_{i+1}-x_i}\frac{Z_i}{U_i}$$

$$\Gamma_{i+1} = \frac{Z_{i+1}}{U_{i+1}}\frac{U_{i+1}-U_i}{x_{i+1}-x_i}$$

$$Z_{i+1} = Z_i + a(x_{i+1}-x_i) - \frac{b(\Gamma_i)}{U_i}(U_{i+1}-U_i)Z_i$$

However, the last term is singular at the stagnation point, $U(0)=0$!

To avoid the division by zero at the first step, we might try the <u>implicit Euler method</u>:

$$\frac{z_{i+1}-z_i}{x_{i+1}-x_i} = a - b_{i+1}\frac{U_{i+1}-U_i}{x_{i+1}-x_i}\frac{z_{i+1}}{U_{i+1}}$$

$$\rho_{i+1} = \frac{z_{i+1}}{U_{i+1}} \cdot \frac{U_{i+1}-U_i}{x_{i+1}-x_i}$$

$$b_{i+1} = \begin{cases} 4.165 & \text{if } \rho_{i+1} \geq 0 \\ 4.579 & \text{if } \rho_{i+1} < 0 \end{cases}$$

$$z_{i+1} + \frac{b_{i+1}}{U_{i+1}}(U_{i+1}-U_i)z_{i+1} = z_i + a(x_{i+1}-x_i)$$

$$z_{i+1} = \frac{z_i + a(x_{i+1}-x_i)}{1+\frac{b_{i+1}}{U_{i+1}}(U_{i+1}-U_i)}$$

Note that the tangential velocity from the outer flow, U, is typically obtained (e.g., by the panel method) at some discrete points x_i.

The shape factor Γ is then obtained from Eq. (8.16), using 1st-order backward finite difference:

$$\Gamma_{i+1} = \frac{Z_{i+1}}{\nu_{i+1}} \frac{U_{i+1} - U_i}{x_{i+1} - x_i} . \qquad (8.33)$$

The parameter β can then be obtained from the implicit relationship

$$\Gamma = \beta \left(B_2(\beta) \right)^2 . \qquad (8.14)$$

The quantities of interest are then obtained as follows:

$$\delta_2 = \left(\frac{Z\,\nu}{U} \right)^{1/2} , \quad \delta_1 = \delta_2 \frac{B_1}{B_2} \qquad (8.25) \; (8.26)$$

$$\delta_N = \frac{\delta_2}{B_2} , \quad \tau_w = \frac{\mu U B_2 f_w''}{\delta_2} \qquad (8.28), (8.29)$$

Dimensionless form of the Walz' integral method:

$$\bar{Z} = \frac{Z}{L} \rightarrow \boxed{\frac{d\bar{Z}}{d\bar{x}} = \frac{dZ}{dx}} \simeq a - b\Gamma$$

$$\Gamma = \frac{\bar{Z}}{\bar{U}}\frac{d\bar{U}}{d\bar{x}} = \frac{Z}{U}\frac{dU}{dx}$$

$$\tilde{\delta}_2 = \frac{\delta_2}{L} = \sqrt{\frac{\bar{Z}}{Re}}$$

$$\bar{\delta}_2 = \frac{\delta_2}{L}\sqrt{Re} = \sqrt{\bar{Z}}$$

$$\bar{\delta}_1 = \frac{\delta_1}{L}\sqrt{Re} = \bar{\delta}_2\frac{\beta_1}{\beta_2}$$

$$c_f = \frac{2\tau_w}{\rho u_\infty^2} = \frac{\mu}{\rho u_\infty L}\sqrt{Re}\,\frac{\bar{U}}{\bar{\delta}_2}\cdot 2\beta_2 f_w''$$

$$= \frac{2}{\sqrt{Re}}\cdot\frac{\bar{U}\beta_2 f_w''}{\bar{\delta}_2}$$

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The computation of the laminar boundary layer should be terminated at the latest when the value $\Gamma = -0.0681$ is reached (flow separation). In practice, the boundary layer often becomes turbulent soon after Γ becomes negative.