

Computation of laminar boundary layer using Hartree profiles

- Follows Chapter 8.1 of Schlichting & Gersten (2017)
- Assume that the velocity profile corresponds locally to the Hartree profile, described by the Falkner-Skan eq. (single parameter β)
- The variation of the boundary-layer thicknesses δ_1 and δ_2 is described by the momentum-integral equation.

The similarity variables are defined as follows:

$$\eta = \frac{y}{\delta_N(x)}, \quad \delta_N = \sqrt{\frac{2\nu x}{U(m+1)}}, \quad m = \frac{\beta}{2-\beta} \quad (8.2)$$

The displacement thickness δ_1 and momentum thickness δ_2 are related to the scaling δ_N as follows:

$$\delta_1 = \beta_1 \delta_N, \quad \beta_1 = \lim_{\eta \rightarrow \infty} (\eta - \xi) \quad (8.3)$$

$$\delta_2 = \beta_2 \delta_N, \quad \beta_2 = \frac{f_w'' - \beta \beta_1}{1 + \beta} \quad (8.4)$$

(8.5)

The wall shear stress is given by: $\frac{\tau_w}{\rho} = \nu \frac{\partial u}{\partial y} \Big|_w = \frac{\nu U}{\delta_N} f_w''$ ①

- Remember that all the quantities β_1, β_2, f_w'' depend on β , because the profile $f(\eta)$ depends on β through the Falkner-Skan equation.
- Substituting (8.3)-(8.5) into (8.2) one obtains (8.7) in Schlichting & Gersten, p. 198:

$$\frac{d(\beta_2 \delta_N)}{dx} + \left(2 + \frac{\beta_1}{\beta_2}\right) \frac{\beta_2 \delta_N}{U} \frac{dU}{dx} = \frac{\nu}{U \delta_N} f_w'' \quad (8.7)$$

- Another relationship between the boundary-layer parameters and the outer pressure gradient is provided by the "compatibility condition at the wall."

Compatibility condition at the wall:

If we express the boundary-layer equation for x -momentum at $y=0$, using the boundary conditions, we obtain the "compatibility condition":

$$\mu \left(\frac{\partial^2 u}{\partial y^2} \right)_w = \frac{dp}{dx} = -U \frac{dU}{dx} \quad (7.2), (8.8)$$

$$\left(\frac{\partial^3 u}{\partial y^3} \right)_w = 0 \quad (7.3)$$

See also chapter 7.1 of S&G, p. 166.

For the Hartree profiles, the compatibility cond. provides additional relationship between param. 2

For the Hartree profiles, the compatibility condition provides the following relationship:

$$f_w''' = -\beta = -\frac{\int_N^2}{V} \frac{dU}{dx} \quad (8.9)$$

see Falkner-Skan

A. Waltz (1966):

Instead of finding an equation for $\beta(x)$, Waltz (1966) defined the following "shape factor":

$$\Gamma = -\frac{\delta_2^2}{U} \left(\frac{\partial^2 U}{\partial y^2} \right)_w = \beta \beta_2^2 \quad (8.13), (8.14)$$

and the following auxiliary variable:

$$Z = \frac{\delta_2^2}{V} U. \quad (8.12)$$

Using the momentum-integral equation and the compatibility condition, one obtains the following system of ODEs for $\Gamma(x)$ and $Z(x)$:

compatibility cond.: $\Gamma = \frac{Z}{U} \frac{dU}{dx} \quad (8.16)$

momentum-integral: $\frac{dZ}{dx} = \underbrace{2\beta_2 f_w''}_{F_1} - \underbrace{\left(3 + 2\frac{\beta_1}{\beta_2} \right) \Gamma}_{H_{12}} \quad (8.15)$

The advantage of introducing the variables Z and Γ is that the RHS of Eq. (8.15) is almost a linear function of Γ . This can be shown by solving the Falkner-Skan eqs. for a range of β and plotting F_2 versus Γ (see Fig. 8.1 on p. 199 of S&G).

Thus, we can replace the RHS of (8.15) with the linear approximation:

(8.20), (8.24)

$$F_2 = a - b\Gamma \quad \text{where} \quad a = 0.441 \quad \text{and} \quad b = \begin{cases} 4.165 & \text{for } \Gamma > 0 \\ 4.579 & \text{for } \Gamma < 0 \end{cases}$$

Momentum-integral eq:

$$\begin{aligned} \Rightarrow \frac{dZ}{dx} &\approx a - b\Gamma = a - b \frac{Z}{U} \frac{dU}{dx} \\ &\approx a - b \frac{Z}{U} \frac{dU}{dx} \end{aligned} \quad (8.21)$$

The variation $Z(x)$ is then obtained by integrating Eq. (8.21), starting from some known initial condition.

At the stagnation point, which is the origin of the b.l., the initial condition

$$Z = 0$$

must be satisfied. We select the stagnation point as the origin $\overset{x=0}{\sqrt{}}$ of the x -coordinate.

Computation of laminar boundary layer

The variation of the shape factor for Hartree profiles can be approximated with the following ODE:

$$\frac{dz}{dx} = a - b(\Gamma) \frac{dU}{dx} \left(\frac{z}{U} \right) \quad (8.21)$$

Exponential growth (blow-up) for $z \frac{dU}{dx} < 0$!

$$\Gamma = \left(\frac{z}{U} \right) \frac{dU}{dx} \quad \text{undefined at } x=0! \quad (8.13)$$

Discretization

$$\text{IC: } z(0) = 0$$

The simplest and cheapest discretization scheme for the initial-value problem is explicit Euler:

$$\frac{z_{i+1} - z_i}{x_{i+1} - x_i} = a - b(\Gamma_i) \frac{U_{i+1} - U_i}{x_{i+1} - x_i} \frac{z_i}{U_i}$$

$$\Gamma_{i+1} = \frac{z_{i+1}}{U_{i+1}} \frac{U_{i+1} - U_i}{x_{i+1} - x_i}$$

$$z_{i+1} = z_i + a(x_{i+1} - x_i) - \frac{b(\Gamma_i)}{U_i} (U_{i+1} - U_i) z_i$$

However, the last term is singular at the stagnation point, $U(0) = 0$!

To avoid the division by zero at the first step, we might try the implicit Euler method:

$$\frac{z_{i+1} - z_i}{x_{i+1} - x_i} = a - b_{i+1} \frac{U_{i+1} - U_i}{x_{i+1} - x_i} \frac{z_{i+1}}{U_{i+1}}$$

$$b_{i+1} = \frac{z_{i+1}}{U_{i+1}} \cdot \frac{U_{i+1} - U_i}{x_{i+1} - x_i}$$

$$b_{i+1} = \begin{cases} 4.165 & \text{if } p_{i+1} \geq 0 \\ 4.579 & \text{if } p_{i+1} < 0 \end{cases}$$

$$z_{i+1} + \frac{b_{i+1}}{U_{i+1}} (U_{i+1} - U_i) z_{i+1} = z_i + a(x_{i+1} - x_i)$$

$$z_{i+1} = \frac{z_i + a(x_{i+1} - x_i)}{1 + \frac{b_{i+1}}{U_{i+1}} (U_{i+1} - U_i)}$$

Note that the tangential velocity from the outer flow, U_i , is typically obtained (e.g., by the panel method) at some discrete points X_i .

$$\rho_{i+1} = \frac{Z_{i+1}}{U_{i+1}} \frac{U_{i+1} - U_i}{X_{i+1} - X_i} \quad (8.33)$$

The parameter β can then be obtained from the implicit relationship

$$\rho = \beta \left(\beta_2(\beta) \right)^2 \quad (8.14)$$

The quantities of interest are then obtained as follows:

$$\delta_2 = \left(\frac{Z U}{U} \right)^{1/2}, \quad \delta_1 = \delta_2 \frac{\beta_1}{\beta_2} \quad (8.25)$$

$$(8.26)$$

$$\delta_N = \frac{\delta_2}{\beta_2}, \quad \chi_w = \frac{\sigma H U \beta_2 f_w''}{2 \delta_2} \quad (8.28), (8.29)$$

Dimensionless form of the Walz' integral method:

$$\bar{z} = \frac{z}{L} \rightarrow \left[\frac{d\bar{z}}{d\bar{x}} = \frac{dz}{dx} \right] \approx a - b\bar{x}$$

$$\bar{\Gamma} = \frac{\bar{z}}{\bar{u}} \frac{d\bar{u}}{d\bar{x}} = \frac{z}{u} \frac{du}{dx}$$

$$\bar{\delta}_2 \equiv \frac{\delta_2}{L} = \sqrt{\frac{\bar{z}}{Re}}$$

$$\bar{\delta}_2 = \frac{\delta_2}{L} \sqrt{Re} = \sqrt{\bar{z}}$$

$$\bar{\delta}_1 \equiv \frac{\delta_1}{L} \sqrt{Re} = \bar{\delta}_2 \frac{\beta_1}{\beta_2}$$

$$C_f = \frac{2\tau_w}{\rho u_\infty^2} = \frac{\mu}{\rho u_\infty L} \sqrt{Re} \frac{\bar{u}}{\bar{\delta}_2} \cdot 2\beta_2 f_w''$$

$$= \frac{2}{\sqrt{Re}} \cdot \frac{\bar{u} \beta_2 f_w''}{\bar{\delta}_2}$$

The computation of the laminar boundary layer should be terminated at the latest when the value $P = -0.0681$ is reached (flow separation). In practice, the boundary layer often becomes turbulent soon after P becomes negative.