# 322.079 Computational Aerodynamics

5.4.2024

## Lecture 4 on 9.4.2024: Subsonic compressible flow around a thin airfoil at zero lift

or supersonic

## A) Governing equations for compressible potential flow:

- Momentum + Continuity:

$$(c^2-\phi_x^2)\phi_{xx} + (c^2-\phi_y^2)\phi_{yy} - 2\phi_x\phi_y\phi_{xy} = 0 \qquad (1)$$

- Total energy + thermodynamic relationships:

$$c^2 = c_\infty^2 + \frac{\varkappa-1}{2}\left(U^2-\phi_x^2-\phi_y^2\right) \qquad (2)$$

where $\nabla\phi = \underline{u}$, $\varkappa = \frac{c_p}{c_v}$

Non-linear eq. → can be solved numerically (FEM / FD)

## B) Dimensionless form: – for simplification & generality

→ define dimensionless variables:

$$\boxed{\tilde{x} = \frac{x}{L},\ \tilde{\underline{u}} = \frac{\underline{u}}{U},\ \tilde{c} = \frac{c}{c_\infty}};\quad \tilde{\phi} = \frac{\phi}{UL} \quad \text{s.t.}\quad \tilde{\underline{u}} = \tilde{\nabla}\tilde{\phi} \qquad (3)$$

→ dimensionless eq.:

$$\left[M_\infty^{-2} + \frac{\varkappa-1}{2}\left(1-\tilde{\phi}_{\tilde{y}}^2\right) - \frac{\varkappa+1}{2}\tilde{\phi}_{\tilde{x}}^2\right]\tilde{\phi}_{\tilde{x}\tilde{x}} + \left[M_\infty^{-2} + \frac{\varkappa-1}{2}\left(1-\tilde{\phi}_{\tilde{x}}^2\right) - \frac{\varkappa+1}{2}\tilde{\phi}_{\tilde{y}}^2\right]\tilde{\phi}_{\tilde{y}\tilde{y}}$$

$$-2\tilde{\phi}_{\tilde{x}}\tilde{\phi}_{\tilde{y}}\tilde{\phi}_{\tilde{x}\tilde{y}} = 0 \qquad (4)$$

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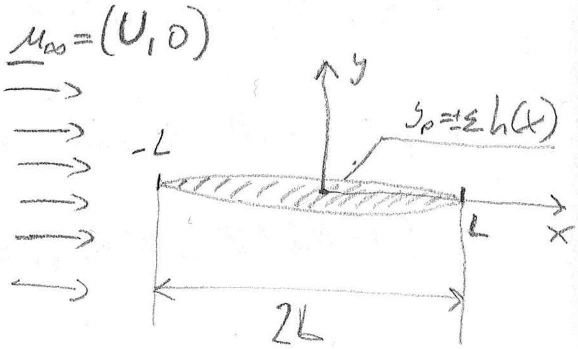
## C) Recall the problem formulation:




- Boundary conditions:

— $\tilde{\underline{u}} \to (1,0)$ as $\tilde{x}^2+\tilde{y}^2 \to \infty$

$\Longrightarrow \tilde{\phi} = \tilde{x}$ as $\tilde{x}^2+\tilde{y}^2 \to \infty$ (5)

"far-field"

— symmetry: $\tilde{\phi}_{\tilde{y}} = 0$ at $\tilde{y}=0$ for $\tilde{x} \le -1$ or $\tilde{x} \ge 1$

(6)

- Kinematic condition:

$$\boxed{\frac{\tilde{v}}{\tilde{u}} = \frac{\tilde{\phi}_{\tilde{y}}}{\tilde{\phi}_{\tilde{x}}} = \pm\varepsilon\frac{\partial\tilde{h}}{\partial\tilde{x}}} \quad \text{at } \tilde{y} = \pm\varepsilon\tilde{h} \text{ for } -1 < \tilde{x} < 1$$

(7)

## D) Asymptotic expansion: (linearization)

- Assumption: the thin profile causes a small perturbation of the incoming flow, proportional to $\varepsilon \ll 1$

$\tilde{\phi} = \phi_0 + \varepsilon\phi_1$ ($\leftarrow$ drop the ~ for brevity) (8)

$\phi_0 = \tilde{x}$ $\leftarrow$ undisturbed flow $\underline{u}_0 = (1,0)$

$\tilde{\phi} = \tilde{x} + \varepsilon\phi_1$

$\tilde{\phi}_{\tilde{x}} = 1 + \varepsilon\phi_{1,\tilde{x}}$

$(\tilde{\phi}_{\tilde{x}})^2 = 1 + 2\varepsilon\phi_{1,\tilde{x}} + \cancel{\varepsilon^2\phi_{1,\tilde{x}}^2}$

$\tilde{\phi}_{\tilde{x}\tilde{x}} = \varepsilon\phi_{1,\tilde{x}\tilde{x}}$

$\tilde{\phi}_{\tilde{y}} = \varepsilon\phi_{1,\tilde{y}}$

$(\tilde{\phi}_{\tilde{y}})^2 = \varepsilon^2\phi_{1,\tilde{y}}^2 \approx 0$

$\tilde{\phi}_{\tilde{y}\tilde{y}} = \varepsilon\phi_{1,\tilde{y}\tilde{y}}$

$\tilde{\phi}_{\tilde{x}\tilde{y}} = \varepsilon\phi_{1,\tilde{x}\tilde{y}}$

$\Rightarrow$ substitute into governing eq. & BCs (drop all ~ for brevity)

## D1) Governing equations:

$$\left[M_\infty^{-2}+\frac{\varkappa-1}{2}-\frac{\varkappa+1}{2}\left(1+\cancel{2\varepsilon\phi_{1,x}}\right)\right]\varepsilon\phi_{1,xx}+\left[M_\infty^{-2}+\cancel{\frac{\varkappa-1}{2}\cdot\left(-2\varepsilon\phi_{1,x}\right)}\right]\varepsilon\phi_{1,yy}-\cancel{2\left(1+\varepsilon\phi_{1,x}\right)\varepsilon^2\phi_{1,y}\phi_{1,xy}}=0$$

- cancel small terms $\varepsilon^2 \lll 1$, $\varepsilon^3 \lll 1$ and divide by $\varepsilon$:

$$\boxed{(1-M_\infty^2)\phi_{1,xx}+\phi_{1,yy}=0}$$

$\leftarrow$ linearized eq. for small disturbances $\varepsilon \ll 1$ (9)

$\Rightarrow$ can be solved analytically
- $\rightarrow M_\infty > 1 \Rightarrow$ wave equation
- $\rightarrow M_\infty < 1 \Rightarrow$ Prandtl-Glauert transformation
  - $\rightarrow$ Laplace eq.
  - $\Rightarrow$ singularity method

## D2) Boundary conditions:

- far field: $\tilde{\phi}=\tilde{x}$ as $\tilde{x}^2+\tilde{y}^2\rightarrow\infty$

$$\cancel{\tilde{x}}+\varepsilon\phi_1=\cancel{\tilde{x}}$$

$$\underline{\phi_1=0} \qquad (10)$$

- symmetry: $\tilde{\phi}_{\tilde{y}}=0$ at $\tilde{y}=0$ for $\tilde{x}\leq-1 \vee \tilde{x}\geq 1$

$$\underline{\cancel{\varepsilon}\,\phi_{1,y}=0} \qquad (11)$$

D2, continuation 

**D2.1** – asymptotic expansion of kinematic BC:

$$\boxed{\phi_y(x,\pm\varepsilon h) = \pm\varepsilon h'\,\phi_x(x,\pm\varepsilon h)} \qquad (7)$$

- Taylor expansion:

$$\phi_y(x,\pm\varepsilon h) = \phi_y(x,0^\pm) \pm \varepsilon h\,\phi_{yy}(x,0^\pm) + \mathcal{O}(\varepsilon^2)$$
$$\phi_x(x,\pm\varepsilon h) = \phi_x(x,0^\pm) \pm \varepsilon h\,\phi_{xy}(x,0^\pm) + \mathcal{O}(\varepsilon^2) \qquad (12)$$

- Substitute asymptotic expansion:

$$\phi_y(x,\pm\varepsilon h) \approx \varepsilon\phi_{1,y}(x,0^\pm) \cancel{\pm\varepsilon h\,\varepsilon\phi_{1,yy}(x,0^\pm)} \rightarrow \varepsilon^2 \ll 1$$
$$\phi_x(x,\pm\varepsilon h) \approx 1 + \varepsilon\phi_{1x}(x,0^\pm) \cancel{\pm\varepsilon h\,\varepsilon\phi_{1,xy}(x,0^\pm)} \qquad (13)$$

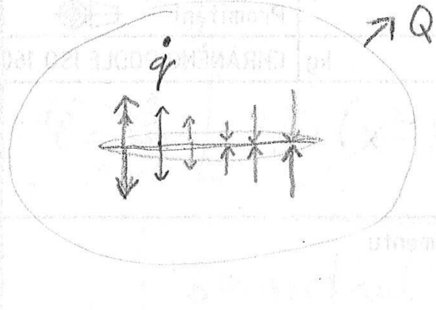
- Substitute (13) into the BC (7):

$$\cancel{\varepsilon}\phi_{1,y}(x,0^\pm) = \pm\cancel{\varepsilon}h'\left[1 + \cancel{\varepsilon\phi_{1x}(x,0^\pm)}\right] \quad /:\varepsilon$$

$$\phi_{1,y}(x,0^\pm) = \pm h' \qquad (14)$$

discontinuous $\frac{\partial\phi_1}{\partial y} = v$ at the airfoil

⇒ For the outer potential flow (excludes thin boundary layer) the airfoil is equivalent to a singular (infinitely thin) source/sink distribution at the axis.




↗ $\dot{Q} = 0$ zero net source ⇒ mass is conserved

$\boxed{\text{Flow described by (9, 10, 11, 14)}}$