4 Compressible flow over a thin airfoil: Asymptotic expansion

In this example, we consider the flow of a compressible fluid over a slender wing profile at zero lift, as introduced in Section 3.2. With the help of asymptotic expansion, we will see how small perturbations of the outer flow can be computed analytically.

Disclaimer: This introductory example follows closely the Chapter 1 of Steinrück (2020).

4.1 Governing equations

For large values of the Reynolds number, $\mathrm{Re} = u_\infty L/\nu \gg 1$, where u_∞ is the velocity of the incoming flow, L is the profile's length and ν is the kinematic viscosity of the fluid, the flow is inviscid except from a thin boundary layer and a wake. The inviscid outer flow is governed by the Euler equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{4.1a}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{4.1b}$$

and the continuity equation

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0\,, \tag{4.1c}$$

where u and v are the velocity components in the x- and y-directions, p is pressure and ρ is the fluid's density. The relationship between pressure and density is governed by additional thermodynamic equations. In general, the pressure $p(\rho, s)$ depends on the density ρ and entropy s. However, for $M_\infty \lesssim 5$ (and in the absence of strong heat sources or heat sinks), the flow can be modelled with good accuracy as isentropic, $s = \mathrm{const}$. Then, $p = p(\rho)$ and the right-hand sides of Eqs. (4.1a) and (4.1b) can be written as

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial \rho}\right)_s\frac{\partial \rho}{\partial x}, \tag{4.2a}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial \rho}\right)_s\frac{\partial \rho}{\partial y}. \tag{4.2b}$$

The derivative of pressure with respect to density at constant entropy defines the local speed of sound c as follows:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s. \tag{4.3}$$

Thus, Eqs. (4.1a) and (4.1b) can be written as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial x}, \tag{4.4a}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial y}. \tag{4.4b}$$

Substituting the expressions for $\partial\rho/\partial x$ and $\partial\rho/\partial y$ from Eqs. (4.4a,b) into the continuity equation (4.1c) gives

$$-\frac{1}{c^2}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)u + \frac{\partial u}{\partial x} - \frac{1}{c^2}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)v + \frac{\partial v}{\partial y} = 0, \tag{4.5a}$$

which can be re-arranged to

$$(c^2 - u^2)\frac{\partial u}{\partial x} - uv\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + (c^2 - v^2)\frac{\partial v}{\partial y} = 0. \tag{4.5b}$$

The local speed of sound in an ideal gas can be expressed with the help of Eq. (4.3) and the equation of state as a function of local temperature T as follows:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \kappa R T, \qquad \text{where} \qquad \kappa = \frac{c_p}{c_v}, \qquad R = c_p - c_v\,, \tag{4.6}$$

and c_p and c_v are the specific heat capacities at constant pressure and constant volume, respectively. Combination of the expressions in Eq. (4.6) gives

$$c^2 = (\kappa - 1) c_p T\,. \tag{4.7}$$

The local temperature T is governed by the conservation of energy. For the isentropic flow, the energy equation reduces to the conservation of total enthalpy:

$$c_p T + \frac{1}{2}(u^2 + v^2) = \text{const.} = c_p T_\infty + \frac{1}{2} u_\infty^2\,, \tag{4.8}$$

where T_∞ is the temperature of the incoming flow far from the wing. Substituting for $c_p T$ in Eq. (4.8) using Eq. (4.7), we obtain an equation for the local speed of sound:

$$\frac{c^2}{\kappa - 1} + \frac{1}{2}(u^2 + v^2) = \frac{c_\infty^2}{\kappa - 1} + \frac{1}{2} u_\infty^2\,. \tag{4.9}$$

The equations (4.5) and (4.9) provide a set of two equations for the unknowns u, v and c. Thus, we need another equation to close the problem. Later we will prove that the inviscid outer flow (in the limit of large Reynolds number) is irrotational, that is, the vorticity is zero

$$\omega = \nabla \times \boldsymbol{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0, \tag{4.10}$$

provided that the incoming flow is irrotational. This is certainly the case when an object moves through a stagnant fluid, or generally when the incoming flow is uniform. Then, Eq. (4.10) serves as additional equation that closes the problem. Thanks to Eq. (4.10) we can introduce a velocity potential φ such that

$$u = \frac{\partial \varphi}{\partial x}, \qquad v = \frac{\partial \varphi}{\partial y}. \tag{4.11}$$

Substituting for u and v from Eq. (4.11) into Eq. (4.10), the Eq. (4.10) is identically satisfied. Substituting also into the equations (4.5) and (4.9) provides a set of two equations

$$(c^2 - \varphi_x^2)\varphi_{xx} - 2\varphi_x \varphi_y \varphi_{xy} + \left(c^2 - \varphi_y^2\right)\varphi_{yy} = 0, \tag{4.12a}$$

$$c^2 = c_\infty^2 + \frac{\kappa - 1}{2}\left(u_\infty^2 - \varphi_x^2 - \varphi_y^2\right) \tag{4.12b}$$

for φ and c. The subscripts in (4.12) indicate partial derivatives. Substituting (4.12b) into (4.12a), one obtains a single equation for the velocity potential φ.

4.2 Dimensionless formulation

For generality, we refer all lengths and velocities to L and u_∞, respectively. Thus, we define the following dimensionless variables:

$$(X, Y) = \frac{(x, y)}{L}, \qquad (U, V) = \frac{(u, v)}{u_\infty}, \qquad \phi = \frac{\varphi}{u_\infty L}, \qquad C = \frac{c}{u_\infty}. \tag{4.13}$$

The equations for the dimensionless velocity potential read

$$(C^2 - \phi_X^2)\phi_{XX} - 2\phi_X \phi_Y \phi_{XY} + (C^2 - \phi_Y^2)\phi_{YY} = 0, \tag{4.14a}$$

$$C^2 = \frac{1}{M_\infty^2} + \frac{\kappa - 1}{2}(1 - \phi_X^2 - \phi_Y^2). \tag{4.14b}$$

4.3 Boundary conditions

The flow over a thin symmetric airfoil at zero angle of attack is sketched in Figure 3. The surface of the airfoil is defined by

$$y = \pm y_p(x) = \pm\epsilon L H(x/L), \tag{4.15}$$

where $\epsilon \ll 1$ is the dimensionless thickness of the profile and $H(X) \in \langle 0,\ 1/2 \rangle$ defines the shape. The surface of the airfoil is an impenetrable barrier for the outer flow. Thus, we require that the outer flow is tangential to the surface:

$$\frac{v}{u} = \pm\epsilon\frac{\mathrm{d}H}{\mathrm{d}X} \qquad \text{at } y = \pm y_p(x) \text{ for } -L/2 \le x \le L/2. \tag{4.16}$$

Since the configuration shown in Figure 3 is symmetric about the horizontal axis $y = 0$, we will consider only the upper half of the flow for $y \ge 0$. In dimensionless form, the no-penetration condition reads

$$\frac{\phi_Y}{\phi_X} = \epsilon\frac{\mathrm{d}H}{\mathrm{d}X} \qquad \text{at } Y = \epsilon H(X) \text{ for } -1/2 \le X \le 1/2\,. \tag{4.17}$$

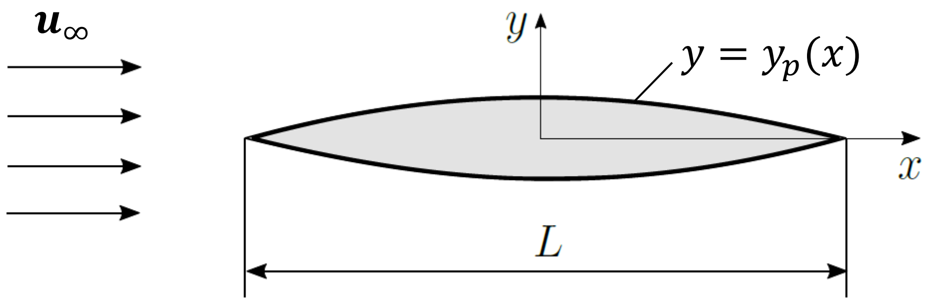

Figure 3: Sketch of the problem.

At the symmetry plane, $Y = 0$, we prescribe the symmetry condition:

$$V \equiv \phi_Y = 0 \qquad \text{at } Y = 0 \text{ for } X < -1/2 \text{ or } X > 1/2. \tag{4.18}$$

Finally, we prescribe the uniform incoming flow with velocity $\boldsymbol{u}_\infty = (u_\infty, 0)$ at large distance from the airfoil. In terms of ϕ, the inflow condition can be written as:

$$\phi = X \qquad \text{as } X \to -\infty. \tag{4.19}$$

The dimensionless velocity potential ϕ of the symmetric outer flow over a given profile with thickness ϵ and shape $H(X)$ is then fully determined by the non-linear equations (4.14) with the boundary conditions (4.17) − (4.19).

4.4 Asymptotic expansion

The asymptotic solution of the Eq. (4.14) with (4.17) – (4.19) for $\epsilon \ll 1$ can be obtained analytically. We take advantage of the fact that we can deduce the solution in the limiting case $\epsilon = 0$. It is the undisturbed flow given by $\phi_0 = X$. We will assume that the thin profile causes a small perturbation of the outer flow, proportional to the thickness ϵ. Thus, we try to write the solution ϕ as a power series with respect to ϵ:

$$\phi = X + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots \tag{4.20}$$

and substitute this Ansatz into the governing equations (4.14). For $\epsilon \ll 1$ we can neglect the *higher-order terms* (h.o.t.) proportional to ϵ^2, ϵ^3, etc., since they are much smaller than ϵ. Thus, we substitute only the two *leading-order terms* on the right-hand side of Eq. (4.20) for ϕ in Eq. (4.14) to obtain

$$\left[C^2 - \left(1 + \epsilon\phi_{1,X}\right)^2\right]\epsilon\phi_{1,XX} - 2\left(1 + \epsilon\phi_{1,X}\right)\epsilon^2\phi_{1,Y}\phi_{1,XY} + \left[C^2 - \epsilon^2\phi_{1,Y}^2\right]\epsilon\phi_{1,YY} + \text{h.o.t.} = 0, \tag{4.21a}$$

$$C^2 = M_\infty^{-2} + \frac{\kappa - 1}{2}\left[1 - \left(1 + \epsilon\phi_{1,X}\right)^2 - \epsilon^2\phi_{1,Y}^2\right] + \text{h.o.t.} \tag{4.21b}$$

Substituting (4.21b) into (4.21a) and neglecting all higher-order terms, we obtain an equation for ϕ_1:

$$(1 - M_\infty^2)\phi_{1,XX} + \phi_{1,YY} = 0\,. \tag{4.22}$$

Note that the Eq. (4.22) for the small flow perturbation ϕ_1 is linear, in contrast to the equation (4.14) for ϕ. For a subsonic flow, $M_\infty < 1$, Eq. (4.22) can be transformed to a Laplace equation (Prandtl-Glauert transformation). For a supersonic flow, $M_\infty > 1$, Eq. (4.22) is a wave equation that can be solved with the D'Alembert solution.

4.5 Expansion of the boundary conditions

The boundary conditions for ϕ_1 can be obtained by substituting the asymptotic expansion (4.20) into the boundary conditions (4.17) – (4.19). Starting with the inflow condition (4.19) we obtain

$$X + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots \to X \qquad \text{as } X \to -\infty. \tag{4.23}$$

Balancing the terms proportional to ϵ on both sides of the equation we obtain

$$\phi_1 \to 0 \qquad \text{as } X \to -\infty. \tag{4.24}$$

That is, the perturbation of the flow must decay towards zero at large distance from the profile, such that the far field is unperturbed.

Similarly, from the symmetry condition (4.18) we obtain

$$\epsilon\phi_{1,Y} + \epsilon^2\phi_{2,Y} + \cdots = 0 \qquad \text{at } Y = 0 \text{ for } X < -1/2 \text{ or } X > 1/2. \tag{4.25}$$

Matching the terms proportional to ϵ leads to

$$\phi_{1,Y} = 0 \qquad \text{at } Y = 0 \text{ for } X < -1/2 \text{ or } X > 1/2. \tag{4.26}$$

The condition (4.17) at the surface of the profile can be simplified for $\epsilon \ll 1$ using Taylor expansion of $\phi_X(X, \epsilon H)$ and $\phi_Y(X, \epsilon H)$ from $Y = 0$ as follows:

$$\phi_X(X, \epsilon H) = \phi_X(X, 0^+) + \epsilon H \phi_{XY}(X, 0^+) + \cdots, \tag{4.27a}$$

$$\phi_Y(X, \epsilon H) = \phi_Y(X, 0^+) + \epsilon H \phi_{YY}(X, 0^+) + \cdots. \tag{4.27b}$$

Substituting the asymptotic expansion (4.20) into the Taylor expansions (4.27) we obtain

$$\phi_X(X, \epsilon H) = 1 + \epsilon\phi_{1,X}(X, 0^+) + \epsilon^2 H\phi_{1,XY}(X, 0^+) + \cdots, \tag{4.28a}$$

$$\phi_Y(X, \epsilon H) = \epsilon\phi_{1,Y}(X, 0^+) + \epsilon^2 H\phi_{1,YY}(X, 0^+) + \cdots. \tag{4.28b}$$

Neglecting higher-order terms, we can write Eq. (4.17) as

$$\epsilon\phi_{1,Y}(X, 0^+) = \epsilon\frac{\mathrm{d}H}{\mathrm{d}X}\left[1 + \epsilon\phi_{1,X}(X, 0^+)\right] \qquad \text{for } -1/2 \le X \le 1/2. \tag{4.29}$$

Matching the same powers of ϵ we obtain

$$\phi_{1,Y}(X, 0^+) = \frac{\mathrm{d}H}{\mathrm{d}X} \qquad \text{for } -1/2 \le X \le 1/2. \tag{4.30}$$

Thus, for $\epsilon \ll 1$ we can enforce the no-penetration condition at the surface of the airfoil by prescribing a vertical velocity along its symmetry axis. With the boundary conditions (4.24), (4.26) and (4.30), the solution ϕ_1 of Eq. (4.22) is fully determined for $Y > 0$. The solution for $Y < 0$ can be obtained as a symmetric reflection about the X-axis. In the following chapters, we will find analytical solutions for ϕ_1 for both subsonic, $M_\infty < 1$, and supersonic, $M_\infty > 1$, flows.

Remark

The no-penetration condition at the lower side of the airfoil

$$\frac{\phi_Y}{\phi_X} = -\epsilon \frac{\mathrm{d}H}{\mathrm{d}X} \qquad \text{at } Y = -\epsilon H(X) \text{ for } -1/2 \leq X \leq 1/2 \tag{4.31}$$

would require

$$\phi_{1,Y}(X, 0^-) = -\frac{\mathrm{d}H}{\mathrm{d}X} \qquad \text{for } -1/2 \leq X \leq 1/2. \tag{4.32}$$

Thus, the no-penetration condition is enforced using a discontinuous vertical velocity inside the airfoil. Eqs. (4.30) and (4.32) are equivalent to distributing a source (for $H' > 0$) or sink (for $H' < 0$) of mass inside the airfoil.

4.6 Velocity, temperature and pressure fields

Remember that the first-order approximation of the flow over the slender profile is defined by the two leading terms in Eq. (4.20). Thus, the dimensionless velocity components are defined, according to Eqs. (4.11) and (4.13), as follows

$$U = \frac{\partial \phi}{\partial X} = 1 + \epsilon \frac{\partial \phi_1}{\partial X}, \qquad V = \frac{\partial \phi}{\partial Y} = \epsilon \frac{\partial \phi_1}{\partial Y}. \tag{4.33}$$

The local temperature and pressure can be obtained from the velocity magnitude $|\boldsymbol{u}|$ through the Bernoulli equation (4.8). It is useful to re-arrange Eq. (4.8) as follows:

$$\begin{aligned} c_p(T - T_\infty) &= \frac{1}{2}(u_\infty^2 - |\boldsymbol{u}|^2), \\ \frac{T - T_\infty}{T_\infty} &= \frac{1}{2}\frac{u_\infty^2 - |\boldsymbol{u}|^2}{c_p T_\infty}. \end{aligned} \tag{4.34}$$

The change of temperature on the left-hand side of Eq. (4.34) is related to the change of pressure through the thermodynamic relationship for the adiabatic compression of an ideal gas (Schneider et al., 2012, p. 85)

$$\frac{\mathrm{d}T}{T} = \frac{\kappa - 1}{\kappa}\frac{\mathrm{d}p}{p}. \tag{4.35}$$

For small disturbances of the incoming flow, $\epsilon \ll 1$, Eq. (4.35) can be simplified as follows:

$$\frac{T - T_\infty}{T_\infty} = \frac{\kappa - 1}{\kappa}\frac{p - p_\infty}{p_\infty}. \tag{4.36}$$

Thus, Eq. (4.36) can be substituted for the left-hand side of Eq. (4.34) to obtain

$$\frac{p - p_\infty}{p_\infty} = \frac{\kappa}{\kappa - 1}\frac{u_\infty^2 - |\boldsymbol{u}|^2}{2 c_p T_\infty}. \tag{4.37}$$

Furthermore, we can use the following relationship for enthalpy

$$h = c_p T = \frac{\kappa}{\kappa - 1}\frac{p}{\rho} \tag{4.38}$$

to substitute for $c_p T_\infty$ in Eq. (4.37) which leads to

$$\begin{aligned} \frac{p - p_\infty}{p_\infty} &= \frac{\rho_\infty}{p_\infty}\frac{u_\infty^2 - |\boldsymbol{u}|^2}{2}, \\ \frac{p - p_\infty}{\rho_\infty u_\infty^2/2} &= \frac{u_\infty^2 - |\boldsymbol{u}|^2}{u_\infty^2}. \end{aligned} \tag{4.39}$$

The left-hand side of Eq. (4.39) is the dimensionless *pressure coefficient*. Eq. (4.39) can be written in terms of dimensionless variables as follows

$$C_p = \frac{p - p_\infty}{\rho_\infty u_\infty^2/2} = 1 - |\boldsymbol{U}|^2. \tag{4.40}$$

4.7 Streamlines

Streamlines of the flow are curves that are at all points tangential to the velocity vector. In steady flows, as considered here, streamlines are the trajectories of infinitesimal fluid parcels transported by the flow. If the coordinates of a fluid parcel are $X_s(t)$ and $Y_s(t)$, then the trajectory is defined by

$$\frac{\mathrm{d}X_s}{\mathrm{d}t} = U\big(X_s(t), Y_s(t)\big), \qquad \frac{\mathrm{d}Y_s}{\mathrm{d}t} = V\big(X_s(t), Y_s(t)\big), \tag{4.41}$$

$$X_s(t) = X_s(t_0) + \int_{t_0}^{t} U\big(X_s(\bar{t}), Y_s(\bar{t})\big)\,\mathrm{d}\bar{t}, \qquad Y_s(t) = Y_s(t_0) + \int_{t_0}^{t} V\big(X_s(\bar{t}), Y_s(\bar{t})\big)\,\mathrm{d}\bar{t}. \tag{4.42}$$

If there is no reversed flow, then the fluid trajectories (streamlines) can be computed in the form $Y_s(X)$ by solving

$$\frac{\mathrm{d}Y_s(X)}{\mathrm{d}X} = \frac{V\big(X, Y_s(X)\big)}{U\big(X, Y_s(X)\big)}, \tag{4.43}$$

$$Y_s(X) = Y_s(X_0) + \int_{X_0}^{X} \frac{V\big(\bar{X}, Y_s(\bar{X})\big)}{U\big(\bar{X}, Y_s(\bar{X})\big)}\,\mathrm{d}\bar{X}\,. \tag{4.44}$$

4.8 Exercises

a) Show yourselves that scaling the variables in Eq. (4.12) according to the definitions in Eq. (4.13), one indeed obtains Eq. (4.14) for the dimensionless variables.

b) Check yourselves that substituting the Eq. (4.21b) into Eq. (4.21a) and neglecting all higher-order terms you obtain the Eq. (4.22).

4.9 Literature

- This chapter mostly overlaps with the chapter 1 of Steinrück (2020).
- For more details about the asymptotic expansion see pp. 59-65 of Schneider (1978).
- For a summary and generalization to three dimensions, see ‘Compressible Potential Flow’, chapter Bon of Brennen (2004).
- The applicability and practical consequences of the equations derived in this chapter are discussed in the book of Prandtl et al. (1993), pp. 125-135.