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Thin airfoil theory

Aim: Compute the incompressible flow over a thin airfoil

Recall:
- Except for a thin boundary layer, the outer flow is potential (irrotational).
- For small Mach number $(M_\infty \lesssim 0.3)$, the flow can be considered incompressible.
- Incompressible potential flow is governed by the Laplace eq.: $\boxed{\Delta \phi = 0}$,

  together with boundary conditions:
  - the outer flow should be tangential to the airfoil surface:

    $$\frac{v}{u} = \frac{\partial Y_s}{\partial x} \quad \text{at} \quad y = Y_s ,$$

  - far from the airfoil, the flow is uniform:

    $$\vec{u} \to \vec{u}_\infty \quad \text{as} \quad x^2 + y^2 \to \infty .$$

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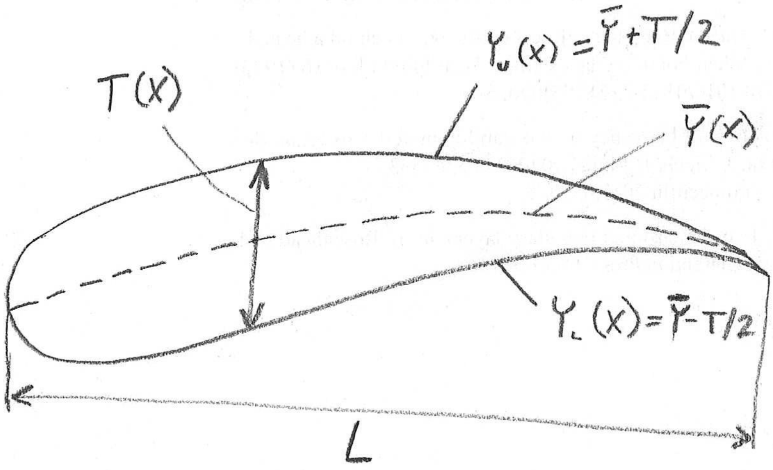
Recap:
- Thin symmetric airfoil at zero angle of attack α (angle between the direction of the incoming flow and the axis of the airfoil) can be represented as a continuous source distribution.
- Lift can be produced with an asymmetric airfoil and/or with a non-zero angle of attack.

Description of the airfoil shape



$$T(x) = Y_u - Y_L$$

$$\bar{Y}(x) = (Y_u - Y_L)/2$$

$\Rightarrow$ Kinematic B.C.:

$$\boxed{\frac{V}{U} = \frac{\partial Y_s}{\partial x} = \bar{Y}' \pm \frac{1}{2} T'} \quad \text{at } Y = \bar{Y} \pm T/2$$

Assumptions:

$T(x) \ll L$,

$\bar{Y}(x) \ll L$,

$\alpha \ll 1$

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Decomposition of the solution:

$$\boxed{\phi = \phi_\infty + \phi_T + \phi_c} \quad \text{where} \quad \phi_T \ll \phi_\infty, \quad \phi_c \ll \phi_\infty$$

and $\phi_T \to 0$, $\phi_c \to 0$ as $x^2+y^2 \to \infty$.

ϕ_∞ is the uniform incoming flow:

$$\phi_\infty = U_\infty (X \cdot \cos\alpha + Y \cdot \sin\alpha).$$

For small $\alpha \ll 1$: $\cos\alpha \approx 1$, $\sin\alpha \approx \alpha$

$$\Rightarrow \boxed{\phi_\infty \approx U_\infty (X + \alpha Y)}$$

Simplified kinematic boundary condition:

$$u(x,y) = \frac{\partial \phi}{\partial x} \approx u_\infty$$

using asymptotic expansion: $\boxed{V(x, 0^{\pm}) \approx u_\infty \left(\bar{Y}' \pm \frac{1}{2} T'\right)}$

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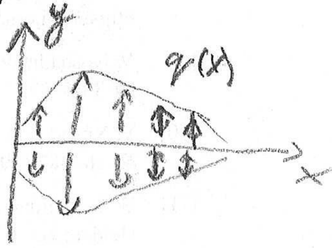
Recall that a symmetric airfoil ($\bar{Y}=0$) at $\alpha=0$ can be represented as a source distribution.

Thus, let ϕ_T satisfy the following B.C:

$$V_T := \frac{\partial \phi_T}{\partial Y} = \pm \frac{1}{2} u_\infty T' \quad \text{at} \quad Y = 0^\pm \quad \text{for} \quad 0 < x < L$$

and $\phi_T \to 0$ as $x^2+y^2 \to \infty$

Satisfy these BCs using a source distribution $q(x)$:



$$\phi_T = \frac{1}{2\pi} \int_0^L q(\bar{x}) \ln\sqrt{(x-\bar{x})^2+y^2}\, d\bar{x}$$

$$u = \frac{\partial \phi_T}{\partial x} = \frac{1}{2\pi} \int_0^L q(\bar{x}) \frac{x-\bar{x}}{(x-\bar{x})^2+y^2}\, d\bar{x}$$

$$v = \frac{\partial \phi_T}{\partial y} = \frac{1}{2\pi} \int_0^L q(\bar{x}) \frac{y}{(x-\bar{x})^2+y^2}\, d\bar{x}$$

Take

$$\lim_{y\to 0^\pm} v = \pm \frac{q(x)}{2} \quad \Rightarrow \quad \boxed{q(x) = 2 v_T(x, 0^+) = u_\infty T'}$$

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Substitute ϕ_∞ and ϕ_T into the Kinematic BC to obtain a B.C. for ϕ_c:

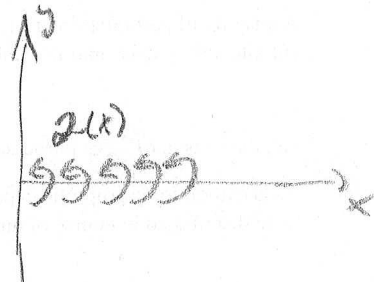
$$V_\infty + V_T + V_c = u_\infty\left(\bar{Y}' \pm \tfrac{1}{2}T'\right) \quad \text{at } Y = 0^\pm \quad \text{for } 0<x<L$$

$$u_\infty \alpha \cancel{\pm \tfrac{1}{2} u_\infty T'} + V_c = u_\infty\left(\bar{Y}' \cancel{\pm \tfrac{1}{2}T'}\right)$$

$$\boxed{V_c = u_\infty\left(\bar{Y}' - \alpha\right)} \quad \text{at } Y = 0 \quad \text{for } 0<x<L$$

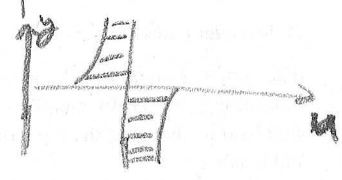
Can be satisfied, e.g., with a potential vortex sheet solution

$$\phi_c = \frac{1}{2\pi}\int_{-\infty}^{\infty} \gamma(\xi)\arctan\frac{y}{x-\xi}\,d\xi$$



$$u_c = \frac{\partial \phi_c}{\partial x} = -\frac{1}{2\pi}\int_0^L \gamma(\xi)\frac{y}{(x-\xi)^2+y^2}\,d\xi$$

$$v_c = \frac{\partial \phi_c}{\partial y} = \frac{1}{2\pi}\int_0^L \gamma(\xi)\frac{x-\xi}{(x-\xi)^2+y^2}\,d\xi$$



$$v_c(x,0) = \frac{1}{2\pi}\,⨍_0^1 \frac{\gamma(\xi)}{x-\xi}\,d\xi \qquad \left(\text{take } L = 1 \text{ and } u_\infty = 1\right)$$

Formal solution for $\gamma(x)$:

$$\gamma(x) = \frac{1}{\sqrt{x(1-x)}}\left[C + \frac{2}{\pi}\int_0^1 \frac{v(\xi,0)}{\xi - x}\sqrt{\xi(1-\xi)}\,d\xi\right]$$

free constant?

Kutta condition

Since the solution for $\gamma(x)$ contains a free constant, there exist ∞ many solutions for ϕ_c!

Note that $\boxed{u_c(x, 0^{\pm}) = \mp \frac{\gamma(x)}{2}}$

$\Rightarrow u_c$ is discontinuous across the airfoil

The Kutta condition states that u is continuous at a sharp trailing edge: $\boxed{u_c(1, 0^+) = u_c(1, 0^-)}$

Thus, $\underline{\gamma(1) = 0}$:

$$C = -\frac{2}{\pi} \int_0^1 \frac{v(\xi, 0)}{\xi - 1} \sqrt{\xi(1-\xi)}\, d\xi$$

$$C = \frac{2}{\pi} \int_0^1 v(\xi, 0) \sqrt{\frac{\xi}{1-\xi}}\, d\xi$$

One can write the solution for $\gamma(x)$ as follows (after some manipulation):

$$\boxed{\gamma(x) = \frac{2}{\pi} \sqrt{\frac{1-x}{x}} \int_0^1 \frac{v(\xi, 0)}{\xi - x} \sqrt{\frac{\xi}{1-\xi}}\, d\xi}$$

With this result we have fully determined ϕ_c and thus, we have found the solution for the flow over the thin airfoil.

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Circulation

The integral of the vortex distribution $\gamma(x)$ gives the total circulation Γ:

$$\Gamma = \int_0^1 \gamma(x)\, dx$$

For any arbitrary closed curve around the airfoil it holds that the following integral over the closed curve equals Γ:

$$\boxed{\oint \vec{u} \cdot \vec{t}\, ds = \Gamma}$$

Bernoulli's equation

$$P + \frac{1}{2}\rho\left(u^2 + v^2\right) = P_\infty + \frac{1}{2}\rho_\infty u_\infty^2 \qquad \text{(dimensional)}$$

$$P - P_\infty = \frac{1}{2}\rho_\infty\left(u_\infty^2 - |\vec{u}|^2\right)$$

$$\frac{P - P_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} = 1 - \frac{|\vec{u}|^2}{u_\infty^2} \qquad \text{(dimensionless)}$$

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Asymptotic expansion of the Bernoulli equation for the thin airfoil theory:

$$|\vec{u}|^2 \approx \left(u_\infty + (u_T + u_c)\right)^2 + \left(\alpha u_\infty + (v_T + v_c)\right)^2$$

$$= u_\infty^2 + 2u_\infty (u_T + u_c) + (u_T + u_c)^2 + \alpha^2 u_\infty^2 + \alpha u_\infty (v_T + v_c) + (v_T + v_c)^2$$

$$\approx u_\infty^2 + 2u_\infty (u_T + u_c)$$

Substitute into the Bernoulli's equation:

$$\boxed{c_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} \approx -2(u_T + u_c)}$$

Kutta - Joukowski Theorem

The force $\vec{F}$ acting on an airfoil in a 2D potential flow is given by:

$$\boxed{\vec{F} = \rho_\infty \vec{u}_\infty \times \vec{\Gamma}}$$

where $|\vec{\Gamma}| = \Gamma$ and the direction of $\vec{\Gamma}$ is given by the sense of rotation of the potential vortex (vortices) through the right-hand rule.

Remark: $\boxed{\vec{F} \perp \vec{u}_\infty}$ $\Rightarrow \vec{F}$ is a lift force

- There is no drag force from the potential flow

- Drag force is created by the boundary layer
- For good airfoils, drag << lift $\Rightarrow \vec{F}$ is close to the real force

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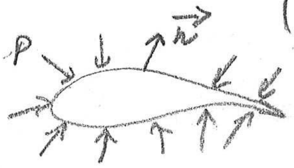
Pressure force on the airfoil.



$$\vec{F} = -\oint_{\{Y_L, Y_u\}} p\,\vec{n}\, ds \quad ; \quad \vec{n} \approx \left(-\frac{dY_s}{dx} ; \pm 1\right)$$

(pressure acts perpendicular to the surface)

Thin airfoil assumptions: $$F_y \approx \int_0^L (p_L - p_u)\, dx \qquad (n_y \approx \pm 1)$$

$$n_x = \vec{n} \cdot \vec{e}_x = -\frac{dY_s}{dx} \qquad F_x = \int_0^L \left(p_u \frac{dY_u}{dx} - p_L \frac{dY_L}{dx}\right) dx$$

$$p_u \approx p_\infty - \rho_\infty u_\infty \left(u_T(x,0) + u_c(x,0^+)\right)$$

$$p_\ell \approx p_\infty - \rho_\infty u_\infty \left(u_T(x,0) + u_c(x,0^-)\right)$$

$$p_u - p_\ell \approx -\rho_\infty u_\infty \left(u_c(x,0^+) - u_c(x,0^-)\right) = -\rho_\infty u_\infty \gamma(x)$$

$$\boxed{F_y \approx \rho_\infty u_\infty \int_0^L \gamma(x)\, dx = \rho_\infty u_\infty \Gamma}$$

Similarly, one can show that

$$F_x \approx -\rho_\infty u_\infty \alpha \Gamma$$