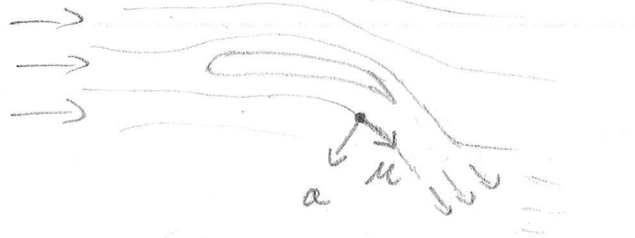
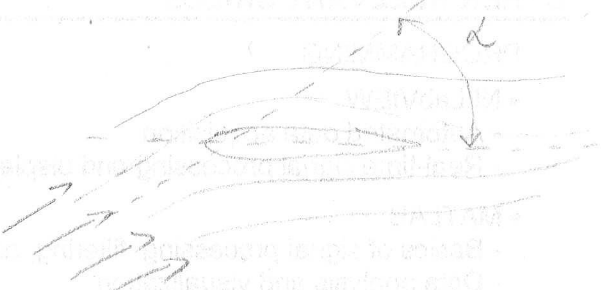
322.079, Lecture 5 (8.4.24)

Incompressible potential flow around thin lifting airfoils

- lift produced by (deflection) of the flow, that is, generated by inertia (Newton's 2nd $\overset{\text{acceleration}}{\text{law}}$ of motion)
- proportional to ρu^2 ; asymmetry around $y=0$ is needed!



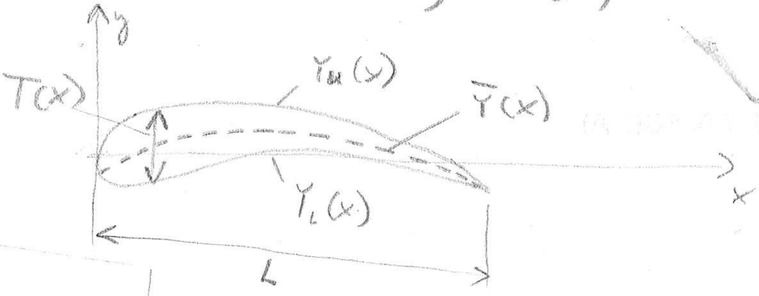
Asymmetry of the profile



Angle of attack

Δ Dimensional variables

- Describe an asymmetric profile with thickness function $T(x)$ and camber function (mean/axis) $\bar{Y}(x)$



$$\boxed{\begin{aligned} T(x) &= Y_u - Y_L \\ \bar{Y}(x) &= \frac{1}{2}(Y_u + Y_L) \end{aligned}}$$

$\Rightarrow$ Kinematic boundary condition

$$\boxed{\frac{v(x, Y_{u,L})}{u(x, Y_{u,L})} = \frac{d\bar{Y}}{dx} \pm \frac{1}{2}\frac{dT}{dx}}$$

- Incompressible potential flow: $M_\infty^2 << 1$

$$\Rightarrow \left[\underline{M_\infty^{-2}} + \frac{\kappa-1}{2}(1-\tilde{\phi}_{\tilde{y}}^2) - \frac{\kappa+1}{2}\tilde{\phi}_{\tilde{x}}^2\right]\tilde{\phi}_{\tilde{x}\tilde{x}} + \left[\underline{M_\infty^{-2}} + \frac{\kappa-1}{2}(1-\tilde{\phi}_{\tilde{x}}^2) - \frac{\kappa+1}{2}\tilde{\phi}_{\tilde{y}}^2\right]\tilde{\phi}_{\tilde{y}\tilde{y}} - 2\tilde{\phi}_{\tilde{x}}\tilde{\phi}_{\tilde{y}}\tilde{\phi}_{\tilde{x}\tilde{y}} = 0$$

$$\boxed{\longrightarrow \tilde{\Delta}\tilde{\phi} = 0 \quad \text{for } M_\infty^2 \to 0}$$

- Recall: $\tilde{\nabla}\tilde{\rho} = M^2 \tilde{\nabla}\tilde{p} \Rightarrow \boxed{\tilde{\rho} = \text{const.}}$ for $M^2 \to 0$

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8.4.24

- Incompressible potential flow described by Laplace eq.

$$\boxed{\Delta \phi = 0}$$

- Since the governing eq. is linear, we can decompose the solution:

$$\phi = \phi_\infty + \overbrace{\phi_T + \phi_c}^{\text{small flow perturbation}}$$

where each component individually satisfies the Laplace equation

- ϕ_∞ - undisturbed flow which satisfies the far-field condition

$$\phi_\infty = U(x \cdot \cos\alpha + y \cdot \sin\underset{\uparrow \text{ angle of attack}}{\alpha}) \Rightarrow \nabla\phi_\infty = (U\cdot\cos\alpha,\ U\cdot\sin\alpha)$$

$\Rightarrow \phi_T \to 0 \wedge \phi_c \to 0$ as $x^2+y^2 \to \infty$

- Again consider small flow perturbation: $\phi_T \ll \phi_\infty$

$\phi_c \ll \phi_\infty$

$\Rightarrow$. small thickness: $T(x) \ll L$

. small angle of attack, $\alpha \ll 1 \Rightarrow \cos\alpha \approx 1$; $\sin\alpha \approx \alpha$

$\Rightarrow \phi_\infty \approx U(x + \alpha y)$

$\nabla\phi_\infty \approx (U, \alpha U)$

. small camber $\overline{Y}(x) \ll L$

again simplify the kinematic condition:

$u(x,y) \approx U$

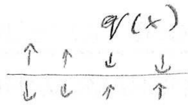
$$\boxed{v(x, 0^\pm) \approx U \cdot \left(\overline{Y}' \pm \frac{1}{2} T'\right)}$$

- ϕ_T - perturbation due to thickness of the airfoil $T(x)$
  - represented by source distribution at $y=0$

- ϕ_T satisfies the boundary condition

$$v_T = \frac{\partial \phi_T}{\partial y} = \pm \frac{1}{2} U T' \quad \text{at} \quad y = 0^{\pm} \quad \text{for} \quad 0 < x < L$$

$\Longrightarrow$ source distribution $\boxed{q(x) = UT'}$



$$v_T = \int_0^L \frac{q(\xi)}{2\pi} \frac{y}{(x-\xi)^2 + y^2} \, d\xi$$

$$u_T = \int_0^L \frac{q(\xi)}{2\pi} \cdot \frac{x-\xi}{(x-\xi)^2 + y^2} \, d\xi$$

$$v_T(x, 0^+) = -v_T(x, 0^-)$$

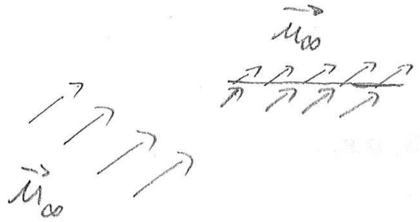
$$\Longrightarrow \quad u_T(x, 0^+) = u_T(x, 0^-)$$

- Determine ϕ_c to satisfy the kinematic BC:

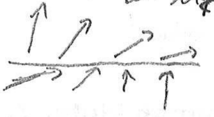
$$U\alpha + \cancel{v_T} + v_c = U\left(\bar{Y}' \pm \cancel{\tfrac{1}{2} T'}\right) \quad \text{at} \quad y = 0^{\pm} \quad \text{for} \quad 0<x<L$$

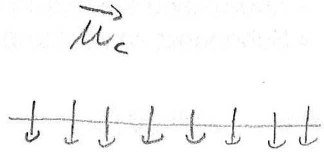
$$v_c = \frac{\partial \phi_c}{\partial y} = U(\bar{Y}' - \alpha)$$

→ compensate net flow through the wing:

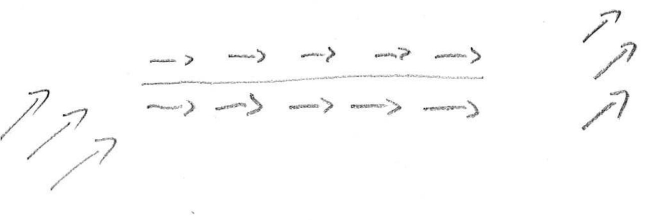


a) $\vec{u}_\infty + \vec{u}_T$

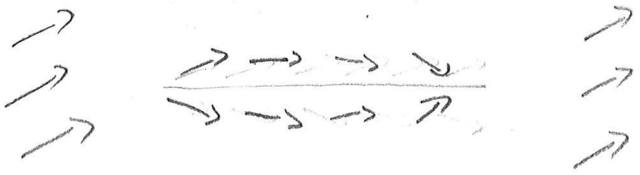




b) $\vec{u}_\infty + \vec{u}_c$



c) $\vec{u}_\infty + \vec{u}_c + \vec{u}_T$



- $\vec{u}_c$ must change the direction of the flow at the airfoil

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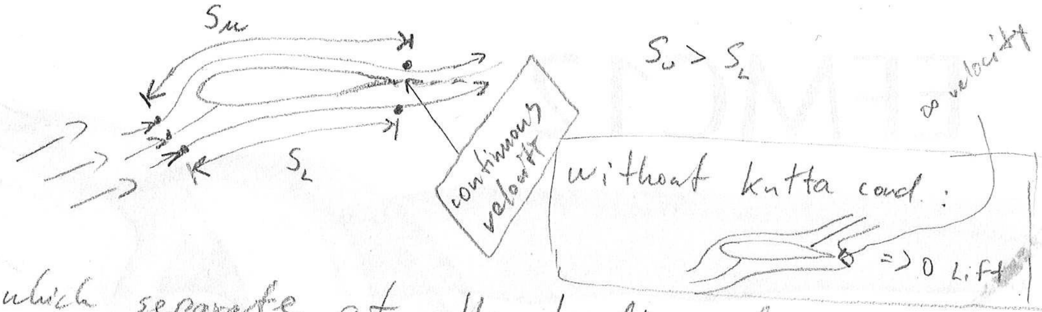
- $\boxed{v_c(x,0^+) = v_c(x,0^-) \; ; \; u_c(x,0^+) \neq u_c(x,0^-)}$

- Kutta condition:

! Requires sharp trailing edge !

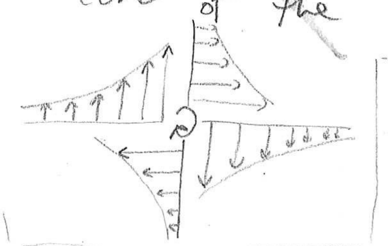


"The two streams which separate at the leading edge join again at the sharp trailing edge, where the flow detaches."

=> ϕ_c can be represented as a potential vortex distribution $\gamma(x)$

Potential vortex - another singularity
(u_c is discontinuous)
- satisfies the potential flow eqs.
=> the flow is still irrotational, (except the singular core of the vortex)

$$\phi_c = -\int_0^L \frac{\gamma(\xi)}{2\pi} \arctan\frac{y}{x-\xi}\, d\xi$$



← Tornado

$$u_c = \int_0^L \frac{\gamma(\xi)}{2\pi} \frac{y}{(x-\xi)^2+y^2}\, d\xi \quad \Rightarrow u_c(x,0^\pm) = \pm\frac{1}{2}\gamma(x)$$

$$v_c = -\int_0^L \frac{\gamma(\xi)}{2\pi} \frac{x-\xi}{(x-\xi)^2+y^2}\, d\xi \quad \Rightarrow v_c(x,0^\pm) = -\fint_0^L \frac{\gamma(\xi)}{x-\xi}\frac{d\xi}{2\pi}$$

! Explain Cauchy Principal Value !

=> substitute into B.C. to determine $\gamma(x)$:

$$v_c(x,0^\pm) = -\frac{1}{2\pi}\fint_0^L \frac{\gamma(\xi)}{x-\xi}\, d\xi = U(Y'-\alpha)$$ ← singular integral eq. for $\gamma(x)$

→ determine $\gamma(x)$ from :

kinematic condition

$$\boxed{\fint_0^L \frac{\gamma(\xi)}{x-\xi}\,d\xi = 2\pi U\left(\alpha - \bar{Y}'(x)\right)}$$

$$\boxed{\begin{array}{l}\text{+ Kutta condition}\\ u(L,0^+) = u(L,0^-)\end{array}} \Rightarrow \boxed{\gamma(L) = 0}$$

• For generality ⇒ dimensionless variables

$$\boxed{\gamma = U\tilde{\gamma}} \qquad \Rightarrow \fint_0^1 \frac{\tilde{\gamma}(\xi)}{\tilde{x}-\xi}\,d\xi = 2\pi\underbrace{\left(\alpha - \tilde{Y}'(\tilde{x})\right)}_{\vartheta(\tilde{x})}$$

⇒ general solution : $\tilde{\gamma}(\tilde{x}) = \frac{1}{\sqrt{\tilde{x}(1-\tilde{x})}}\left[C + \frac{2}{\pi}\fint_0^1 \frac{\vartheta(\xi)}{\xi - \tilde{x}}\sqrt{\xi(1-\xi)}\,d\xi\right]$

! any value of C solves the original problem

⇒ C determined by the Kutta condition

$$\boxed{\tilde{\gamma}(1) = 0} \quad \Rightarrow C = -\frac{2}{\pi}\fint_0^1 \frac{\vartheta(\xi)}{\xi - 1}\sqrt{\xi(1-\xi)}\,d\xi = \frac{2}{\pi}\fint_0^1 \vartheta(\xi)\sqrt{\frac{\xi}{1-\xi}}\,d\xi$$

$$\boxed{\Rightarrow \tilde{\gamma}(\tilde{x}) = \frac{2}{\pi}\sqrt{\frac{1-\tilde{x}}{\tilde{x}}}\fint_0^1 \frac{\vartheta(\xi)}{\xi-\tilde{x}}\sqrt{\frac{\xi}{1-\xi}}\,d\xi}$$

Examples:

1) Symmetric airfoil at Angle of Attack

→ either $\boxed{\alpha \neq 0}$ or $\tilde{Y}' = \text{const.} \neq 0$

$$\tilde{\gamma}(\tilde{x}) = \frac{2\alpha}{\pi} \sqrt{\frac{1-\tilde{x}}{\tilde{x}}} \underbrace{\fint_0^1 \frac{1}{\xi - \tilde{x}} \sqrt{\frac{\xi}{1-\xi}}\, d\xi}_{\pi} = 2\alpha \sqrt{\frac{1-\tilde{x}}{\tilde{x}}}$$

$$\tilde{\Phi} = \tilde{\Phi}_\infty + \tilde{\Phi}_c \qquad \tilde{\Phi}_\infty = \tilde{x} + \alpha \tilde{y}$$

$$\tilde{\Phi}_c = -\int_0^1 \frac{\alpha}{\pi} \sqrt{\frac{1-\xi}{\xi}} \arctan \frac{\tilde{y}}{\tilde{x} - \xi}\, d\xi$$

$$\tilde{u}_c = \frac{\alpha}{\pi} \int_0^1 \sqrt{\frac{1-\xi}{\xi}} \frac{y}{(x-\xi)^2 + y^2}\, d\xi$$

$$\tilde{v}_c = -\frac{\alpha}{\pi} \int_0^1 \sqrt{\frac{1-\xi}{\xi}} \frac{x-\xi}{(x-\xi)^2 + y^2}\, d\xi$$

a) tangential velocity at plate surface

$$\tilde{u}_c(\tilde{x}, 0^\pm) = \pm \alpha \sqrt{\frac{1-\tilde{x}}{\tilde{x}}}$$

$$\tilde{v}_c(\tilde{x}, 0) = -\frac{\alpha}{\pi} \int_0^1 \sqrt{\frac{1-\xi}{\xi}} \cdot \frac{1}{x-\xi}\, d\xi = -\alpha \; \overset{!}{0}$$

Bernoulli equation

b)

$$p + \frac{1}{2}\rho |u|^2 = p_\infty + \frac{1}{2}\rho_\infty U^2$$

$$p - p_\infty = \frac{1}{2}\rho_\infty (U^2 - |u|^2) \approx -\rho_\infty U (u_T + u_c)$$

$$\frac{p - p_\infty}{\frac{1}{2}\rho_\infty U^2} = 1 - |\tilde{u}|^2 \approx -2\rho_\infty |\tilde{u}_c + \tilde{u}_T|$$

Analytical integrals:

$$\int_0^1 \sqrt{\frac{1-\xi}{\xi}} \frac{y}{(x-\xi)^2 + y^2} d\xi$$

$$= \frac{\pi}{2}\left(\sqrt{\frac{z-1}{z}} + \sqrt{\frac{\bar{z}-1}{\bar{z}}}\right)$$

??

$$\int_0^1 \sqrt{\frac{1-\xi}{\xi}} \cdot \frac{1}{x-\xi} d\xi = \pi - \pi\sqrt{\frac{x-1}{x}}$$

Forces acting on the airfoil:

$$\boxed{F_y = \int_0^L (p_L - p_u)\, dx}$$

a)

! Force per unit span

c)

$$C_L = \frac{\int (p - p_\infty)}{\frac{1}{2}\rho_\infty U^2 L} \approx 2 \underbrace{\int_0^1 \tilde{\gamma}(\tilde{x})\, d\tilde{x}}_{\tilde{\Gamma}}$$

"circulation"

d)

$$\int_0^1 \sqrt{\frac{1-\tilde{x}}{\tilde{x}}}\, d\tilde{x} = \frac{\pi}{2}$$

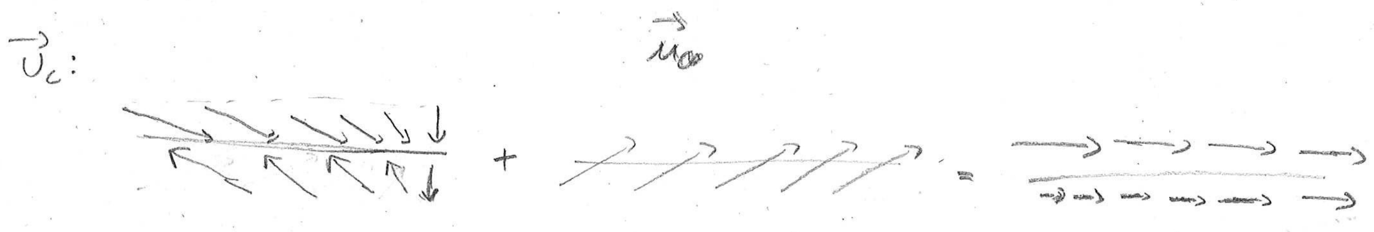
$$\int_0^1 \sqrt{\frac{1-\xi}{\xi}} \cdot \frac{1}{x-\xi} d\xi = \pi$$

for $0 < x < 1$

e)

$$\boxed{C_L \approx 2\pi\alpha}$$

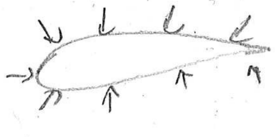
f)



For a thin airfoil $(T(x) \neq 0 \rightarrow \phi_T \neq 0)$: Example 2

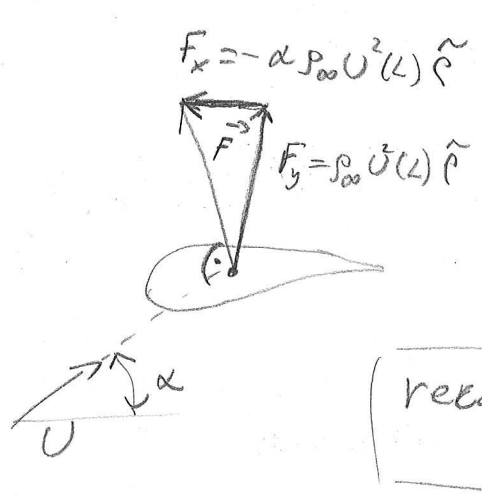
$$F_x = \int_0^L \left(p_u \frac{dy_u}{dx} - p_l \frac{dy_l}{dx}\right)$$

* Pressure force perpendicular to surface :



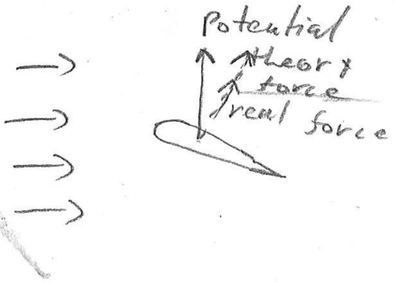
Horizontal force after simplification (p 86-87 of Moran)

$$\frac{F_x}{\rho_\infty U^2 L} \approx -\alpha \int_0^1 \tilde{\gamma}(\tilde{x})\, d\tilde{x} = -\alpha \tilde{\Gamma}$$



recall $\alpha << 1$
$\tan \alpha \approx \alpha$

! In potential flow theory (friction is neglected), the net force is always perpendicular to the free stream (drag force is zero)

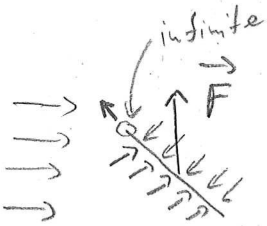


⇒ Kutta-Joukowski Theorem

$$\vec{F} = \rho_\infty \vec{U} \times \vec{\Gamma}$$

right-hand rule

⇒ Paradox:
for infinitely thin plate: $T(x)=0$



Leading-edge suction