7 Singularity method

7.1 Incompressible potential flows

Flows over airfoils at low Mach numbers ($M_\infty \lesssim 0.3$) can be assumed incompressible, that is, the density of the fluid can be assumed constant, ρ = const. Then, the continuity equation (4.1c) reduces to

$$\nabla \cdot \boldsymbol{u} = 0\,. \tag{7.1}$$

If the flow is also irrotational,

$$\nabla \times \boldsymbol{u} = 0, \tag{7.2}$$

which is the case outside of the thin boundary layers, we can define the velocity vector field as the gradient of the velocity potential,

$$\boldsymbol{u} = \nabla\varphi\,. \tag{7.3}$$

Substituting (7.2) into (7.1) we observe that the velocity potential is governed by the Laplace equation

$$\Delta\varphi = 0\,. \tag{7.4}$$

For two-dimensional incompressible flows, it is useful to define also another scalar field, the stream function ψ, such that the velocity vector is everywhere perpendicular to the gradient of ψ:

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}. \tag{7.5}$$

In other words, the isolines of ψ are the streamlines of the flow. Note that when the velocity field is defined in terms of the stream function, as in Eq. (7.5), then the continuity equation (7.1) is identically satisfied. On the other hand, substituting the definition (7.5) into Eq. (7.2), we find that the stream function ψ is as well governed by a Laplace equation

$$\Delta\psi = 0\,. \tag{7.6}$$

Of course, a given flow problem imposes different boundary conditions on φ and ψ. Thus, depending on the flow problem, it may be convenient to search for solutions either for φ or for ψ. Also, for any incompressible potential flow solution, there exists another solution with perpendicular velocity vectors of the same magnitude. Such complementary solution can be obtained, e.g., by exchanging the definitions of φ and ψ (or of the velocity components), and changing the sign of one of them. Of course, the velocity components are also exchanged in the boundary conditions satisfied by the two solutions.

Due to the linearity of (7.4) and (7.6), a linear combination of solutions is also a solution. Furthermore, the solutions can be scaled, translated and rotated arbitrarily. The idea of the singularity method, covered in this chapter, is to obtain two-dimensional incompressible potential flows over different profiles by a superposition of certain elementary analytical solutions. This aim is achieved when one streamline of the flow coincides with the contour of the profile. For some simple shapes, the singularity method provides exact analytical solutions for the flow as a superposition of a few elementary solutions. For flows over more complicated profiles, the same approach can provide approximate numerical solutions as a superposition of many elementary potential flows. Such numerical approach is called the *Panel Method*.

In this chapter, we will introduce the most common elementary potential flows employed within the singularity method and show how they can be used to construct flows over canonical profiles. Afterward, we will apply the singularity method to flows over thin airfoils.

7.2 Elementary solutions

Uniform flow

In the example in chapter 4 we have already encountered the simplest potential flow, the uniform flow, as the inflow condition. Uniform flow is any flow where the velocity vector does not vary in space. When the uniform flow is aligned with the *x*-axis, it is described by

$$\boldsymbol{u} = (u_\infty, 0), \qquad \varphi = u_\infty X, \qquad \psi = u_\infty Y. \tag{7.7}$$

Of course, any potential flow solution, as that in Eq. (7.7), can be multiplied with an arbitrary constant to obtain the desired velocity magnitude. If we exchange the definition of $-\varphi$ and ψ in Eq. (7.7), we obtain a vertical uniform flow (aligned with the *y*-axis). If the uniform flow is inclined with respect to the *x*-axis by some arbitrary angle α, then it is described by

$$\boldsymbol{u} = (u_\infty \cos\alpha\,, u_\infty \sin\alpha), \qquad \varphi = u_\infty (X\cos\alpha + Y\sin\alpha), \qquad \psi = u_\infty (Y\cos\alpha - X\sin\alpha). \tag{7.8}$$

Source flow

Another important solution is the *source flow*. It is a flow which originates from an infinitely small source (or sink) of a finite mass flux, ρQ. The velocity vector points in the radial direction away from (or towards) the source, that is, the streamlines are straight lines originating from the source. Furthermore, the velocity magnitude depends only on the radial distance from the source, *r*. A three-dimensional source flow originates from a single point. A two-dimensional source flow originates from a line perpendicular to the *x*-*y* plane, see Figure 6. You can imagine the line source as a porous hose that discharges fluid through the pores uniformly in all directions with a volume flow rate Q per unit length of the hose.

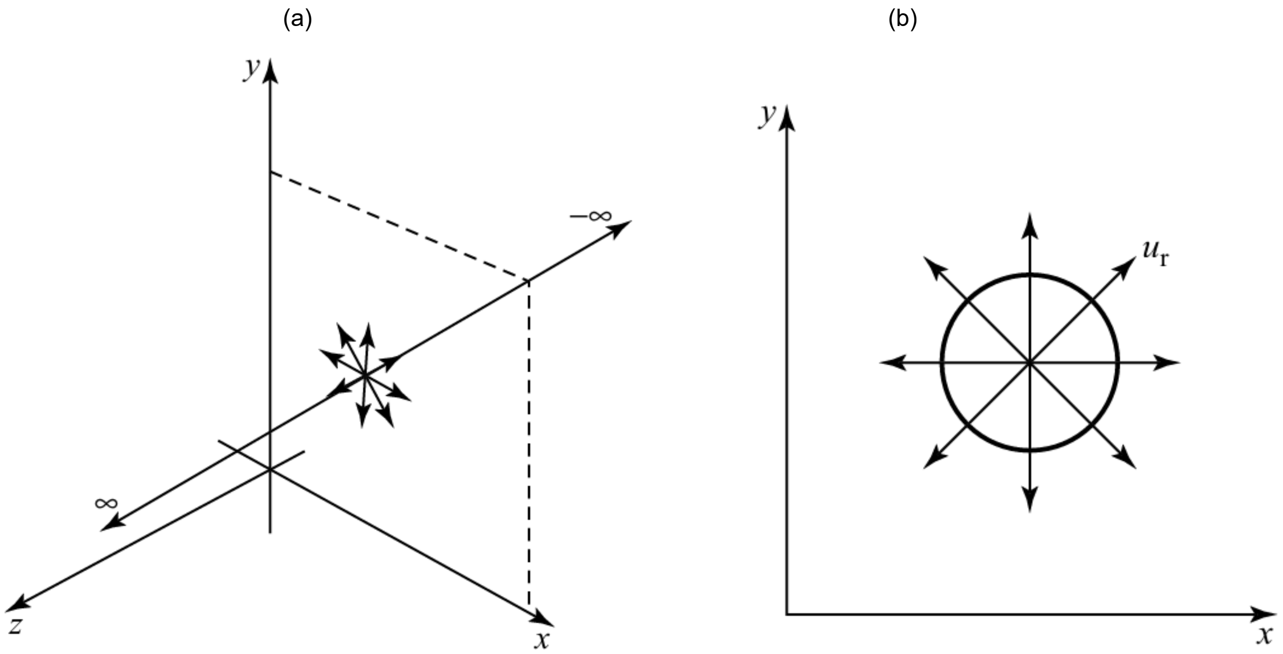

Figure 6: Line source. Image taken from Flandro et al. (2011), p. 142.

The two-dimensional lines-source flow emanating from the *z*-axis (origin in the *x*-*y* plane) takes a simple form in polar coordinates: $\boldsymbol{u} = u_r(r)\boldsymbol{e}_r$, where $r = \sqrt{x^2 + y^2}$ is the radial distance from the origin. Thus, the velocity potential varies only in r-direction, $\varphi = \varphi(r)$. The Laplace equation for φ in polar coordinates then simplifies to an ODE

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\varphi}{\mathrm{d}r} = 0 \tag{7.9}$$

with the solution

$$\varphi = C\ln r = C\ln\sqrt{x^2 + y^2}\,. \tag{7.10}$$

The radial velocity component is then given by

$$u_r = \frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{C}{r}. \tag{7.11}$$

The same dependence $u_r \propto 1/r$ could be obtained from the overall mass conservation. For an incompressible flow, the volume flux Q supplied by the line source per unit length must equal the total volume flux through the surface of a cylinder of unit length and arbitrary radius r around the line source:

$$Q = 2\pi r u_r \,, \tag{7.12}$$

$$u_r = \frac{Q}{2\pi r} \,. \tag{7.13}$$

Comparing the results (7.11) and (7.13) we find that the constant *C* in (7.10) and (7.11) is related to the volume flux Q through the overall mass balance (7.12) as

$$C = \frac{Q}{2\pi} \,. \tag{7.14}$$

Thus, the solution for the velocity potential can be written as

$$\varphi = \frac{Q}{2\pi} \ln\sqrt{x^2 + y^2} \,. \tag{7.15}$$

For $Q < 0$, the direction of the flow is reversed (as compared to $Q > 0$), and the source turns into a sink. The stream function of the source (or sink) flow is given by

$$\psi = \frac{Q}{2\pi} \tan^{-1}\frac{y}{x} \,. \tag{7.16}$$

Note that (7.15) is the Green's function of the two-dimensional Laplace equation (7.4). Thus, it satisfies the continuity equation (7.1) everywhere except from the origin. At the origin, the radial velocity is infinite (singular). This will, however, not be a problem, because we will use this solution only to construct outer flows at some distance from the source.

Potential vortex

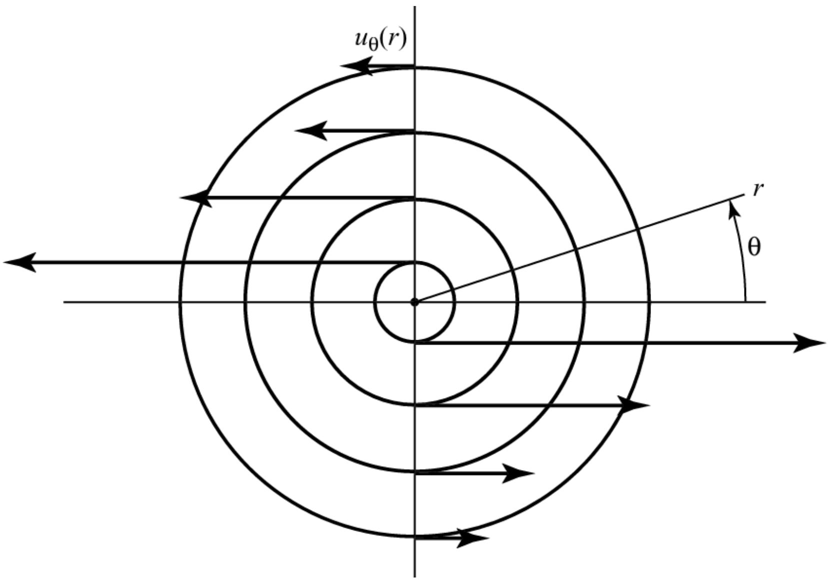

Figure 7: Potential vortex at the origin of a polar coordinate system. Taken from Flandro et al. (2011).

If we exchange the expressions for the velocity potential (7.15) and the stream function (7.16) as follows

$$\varphi = \frac{\Gamma}{2\pi} \tan^{-1}\frac{y}{x} \,, \tag{7.17}$$

$$\psi = -\frac{\Gamma}{2\pi} \ln\sqrt{x^2 + y^2} \,, \tag{7.18}$$

we obtain a flow that is perpendicular to the source flow, see Figure 7. This flow is called the *potential vortex*. The constant Γ, called *circulation*, quantifies the strength of the vortex. In polar coordinates (r, θ), the velocity vector takes the following form

$$\boldsymbol{u} = u_\theta(r)\boldsymbol{e}_\theta \,, \tag{7.19}$$

$$u_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} = \frac{\Gamma}{2\pi}\frac{1}{r}.$$

The azimuthal velocity component u_θ is singular at the center of the vortex. For any closed curve enclosing the center of the vortex, the following equality holds:

$$\oint \boldsymbol{u}\cdot\boldsymbol{t}\,\mathrm{d}s = \Gamma\,, \tag{7.20}$$

where $\boldsymbol{t}$ is the unit tangent vector to the curve and s is the arc length along the curve.

Note that potential vortex flow, described by Eqs. (7.17)-(7.19), is still irrotational (everywhere except from the center of the vortex), that is, it satisfies Eq. (7.2). This means that fluid elements transported by the potential vortex flow do not rotate around their own centers as they orbit about the center of the vortex.

7.3 Superposition of solutions

Source-sink pair

Consider one source and one sink, both of the same strengths $\pm Q$, located at $\boldsymbol{X} = (-a, 0)$ and $\boldsymbol{X} = (a, 0)$, respectively. The velocity potential of the flow induced by this source-sink pair is obtained by superposition as

$$\varphi = \frac{Q}{2\pi}\ln r_1 - \frac{Q}{2\pi}\ln r_2 = \frac{Q}{2\pi}\ln\frac{r_1}{r_2}, \tag{7.21}$$

where

$$r_{1,2} = \sqrt{(x \pm a)^2 + y^2} \tag{7.22}$$

are the distances of an arbitrary evaluation point $P\ (x, y)$ from the source and the sink, respectively, as sketched in Figure 8a. On the other hand, the stream function is defined in terms of the angle of view $\Delta\theta$ of the source-sink pair from the point *P*, illustrated in Figure 8b, as follows

$$\psi = \frac{Q}{2\pi}\,\theta_1 - \frac{Q}{2\pi}\,\theta_2 = -\frac{Q}{2\pi}\,\Delta\theta\,. \tag{7.23}$$

(a) (b)

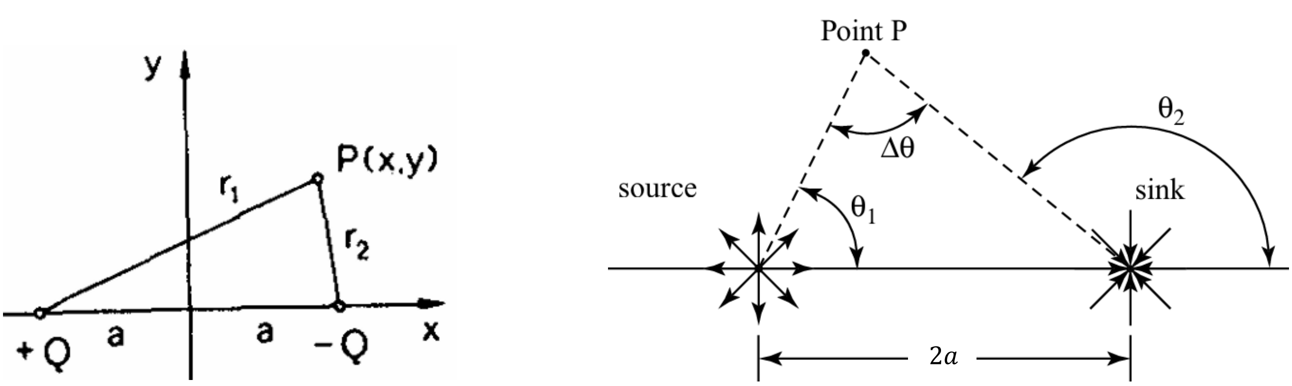

Figure 8: Sketch of a source-sink pair. (a) taken from Schneider (1978), (b) taken from Flandro et al. (2011).

The flow induced by the source-sink pair is shown in Figure 9a. When the pair is located in a uniform flow parallel to the line sector connecting the source and the sink, we obtain the flow over a Rankine oval (Figure 9b).

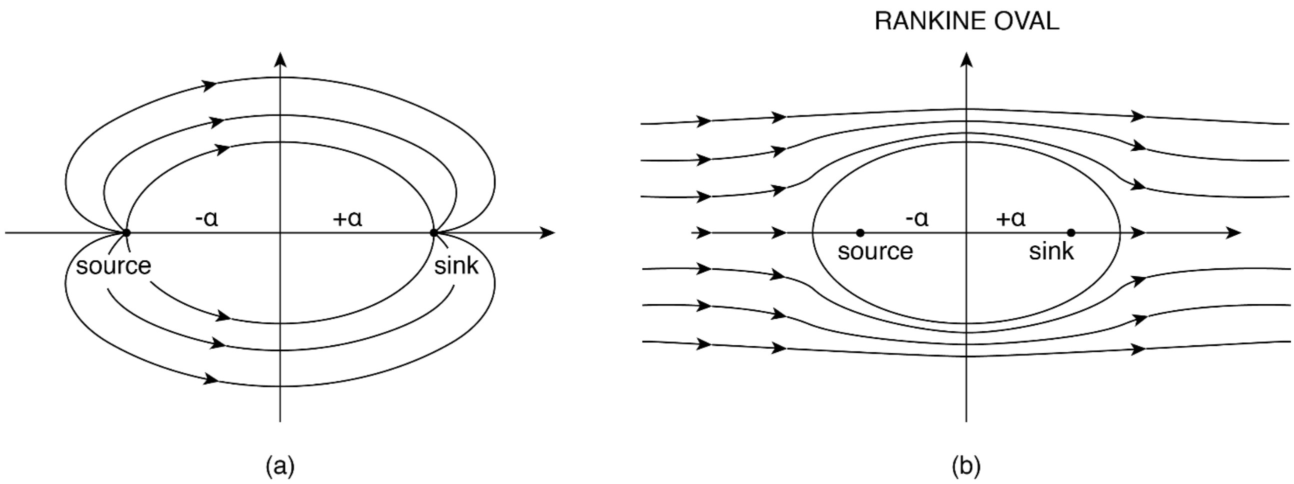

Figure 9: The flow induced by a source-sink pair in an otherwise stagnat fluid (a) and in a uniform flow (b). Taken from Liburdy (2021)

Dipole

Consider the flow over the source-sink pair, described above, in the limit of vanishing distance between the source and the sink, $a \to 0$, such that the product $M = 2aQ =$ const. Taking this limit of the expression (7.21) with (7.22) we obtain the following velocity potential:

$$\varphi = \lim_{\substack{a\to 0\\ 2aQ=M}} \frac{Q}{2\pi} \ln \sqrt{\frac{(x+a)^2+y^2}{(x-a)^2+y^2}} = \frac{M}{2\pi}\frac{x}{x^2+y^2}\,. \tag{7.24}$$

The stream function can be obtained analogously as

$$\psi = \frac{M}{2\pi}\frac{y}{x^2+y^2} \tag{7.24}$$

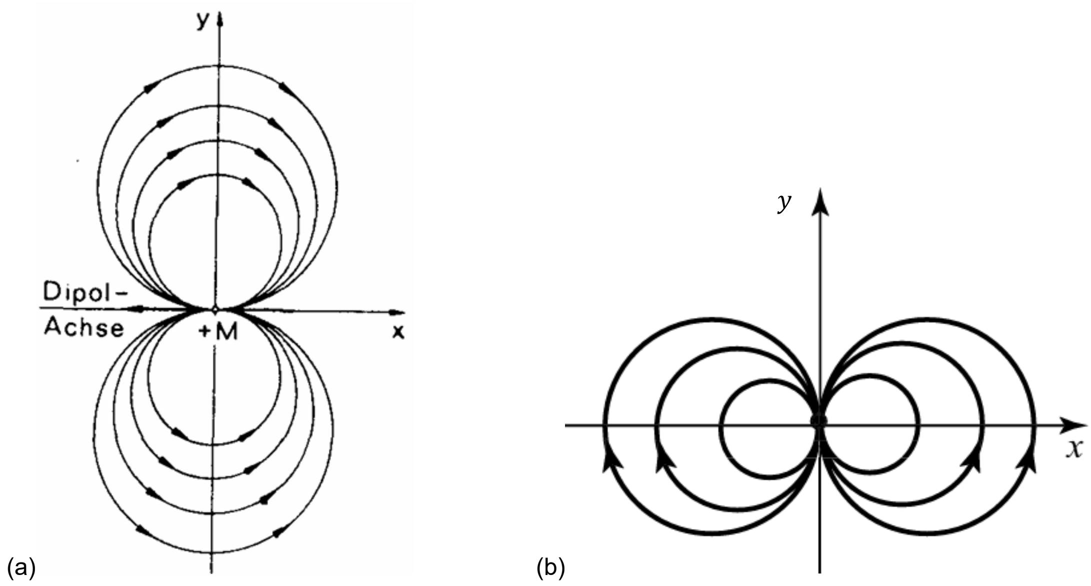

Figure 10: Horizontal (a) and vertical (b) dipole. Taken from Schneider (1978) and Flandro et al. (2011).

This solution is called the (plane) *dipole*. The streamlines of the flow around a dipole are shown in Figure 10a. Note that the dipole has a certain orientation, since the right side of the dipole is a sink and the left side is a source. The axis of the dipole goes from the sink side to the source side, that is, in the negative *x*-direction. If we exchange the definition of φ and

ψ, we obtain a vertical dipole, as shown in Figure 10b. Any other orientation can be obtained by a change of coordinates. When the dipole solution is superposed with a uniform flow in the opposite direction to the axis of the dipole, we obtain a flow over a cylinder, as shown in Figure 11a. If we also superpose a potential vortex in the centre of the cylinder, we obtain an asymmetric flow over the cylinder with circulation, as shown in Figure 11b.

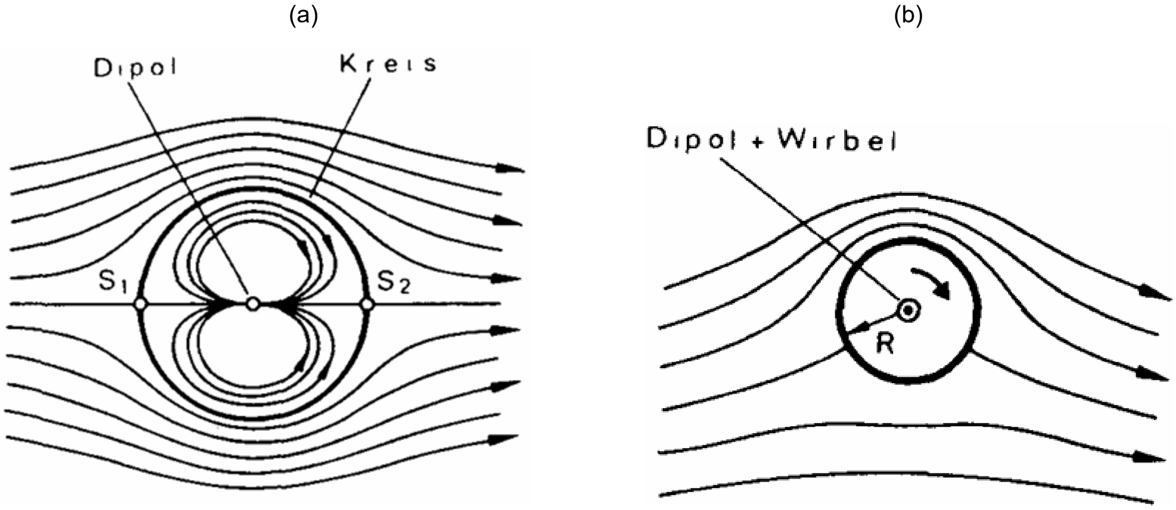

Figure 11: Plane dipole (and in (b) also a vortex) in a parallel flow, leading to a flow over a cylinder (a) without and (b) with circulation. Taken from Schneider (1978).

7.4 Continuous distributions of singularities

The singularities introduced above (source, vortex, dipole) can be also distributed continuously along a given curve, surface, or volume.

Plane source

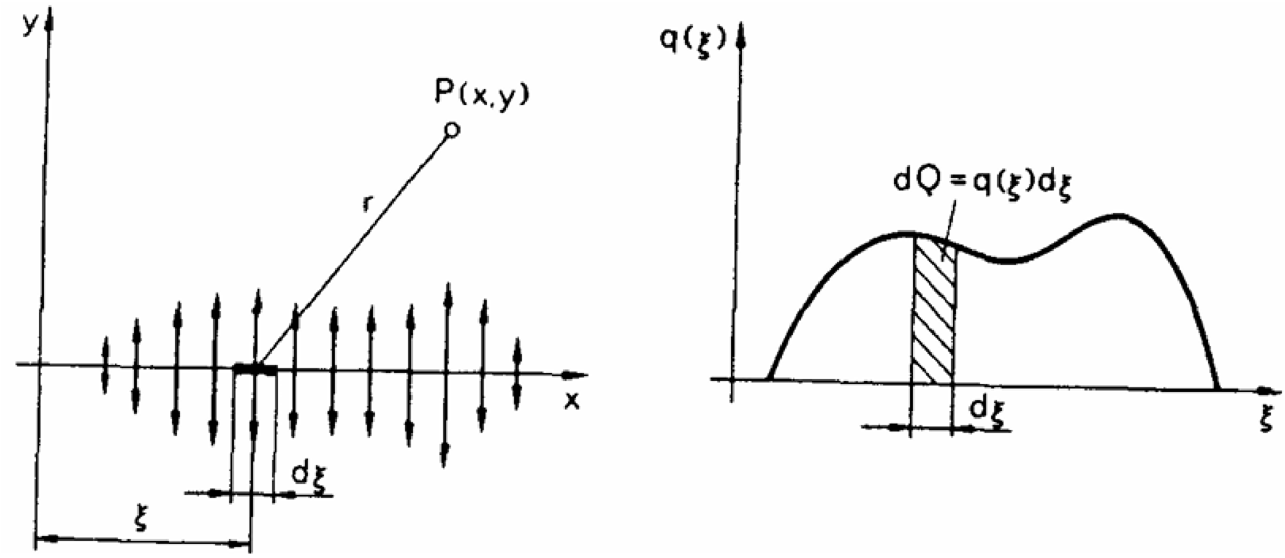

Figure 12: Continuous source distribution on the *x*-axis. Taken from Schneider (1978).

As an example, let us consider a continuous source distribution $q(x)$ along the *x*-axis, as shown in Figure 12. Knowing that the line-source solution (7.15) is the Green's function of Eq. (7.4), the solution for the flow near the continuous source sheet can be obtained as follows:

$$\varphi = \frac{1}{2\pi} \int_{-\infty}^{\infty} q(\xi) \ln \sqrt{(x-\xi)^2 + y^2} \, \mathrm{d}\xi \, . \tag{7.25}$$

Analogously, the stream function of the flow is given by

$$\psi = \frac{1}{2\pi} \int_{-\infty}^{\infty} q(\xi) \tan^{-1} \frac{y}{x-\xi} \, \mathrm{d}\xi \, . \tag{7.26}$$

The velocity components at an arbitrary point (x, y) are obtained as follows:

$$u = \frac{\partial \varphi}{\partial x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} q(\xi) \frac{x-\xi}{(x-\xi)^2 + y^2} \, \mathrm{d}\xi \, , \tag{7.27}$$

$$v = \frac{\partial \varphi}{\partial y} = \frac{1}{2\pi} \int_{-\infty}^{\infty} q(\xi) \frac{y}{(x-\xi)^2 + y^2} \, \mathrm{d}\xi \, . \tag{7.28}$$

The vertical velocity at the *x*-axis, where the source is distributed, can be obtained by taking the limit of Eq. (7.28) as $y \to 0^{\pm}$. Let us consider the case when the source is distributed over a finite segment of the *x*-axis, $x \in \langle 0, 1 \rangle$. Then,

$$\begin{aligned} v(x, 0^{\pm}) &= \frac{1}{2\pi} \lim_{y \to 0^{\pm}} \int_0^1 q(\xi) \frac{y}{(x-\xi)^2 + y^2} \, \mathrm{d}\xi \\ &= \frac{q(x)}{2\pi} \lim_{y \to 0^{\pm}} \int_0^1 \frac{y}{(x-\xi)^2 + y^2} \, \mathrm{d}\xi \\ &= \pm \frac{q(x)}{2} . \end{aligned} \tag{7.29}$$

Thus, the source distribution is determined by the boundary conditions for v,

$$q(x) = v(x, 0^+) - v(x, 0^-) = 2v(x, 0^+). \tag{7.30}$$

Note that the vertical velocity is discontinuous across the plane source, but it is not singular, in contrast to the line source. A plane source in a parallel uniform flow can be used to represent the flow over a thin symmetric profile at zero angle of attack.

Vortex sheet

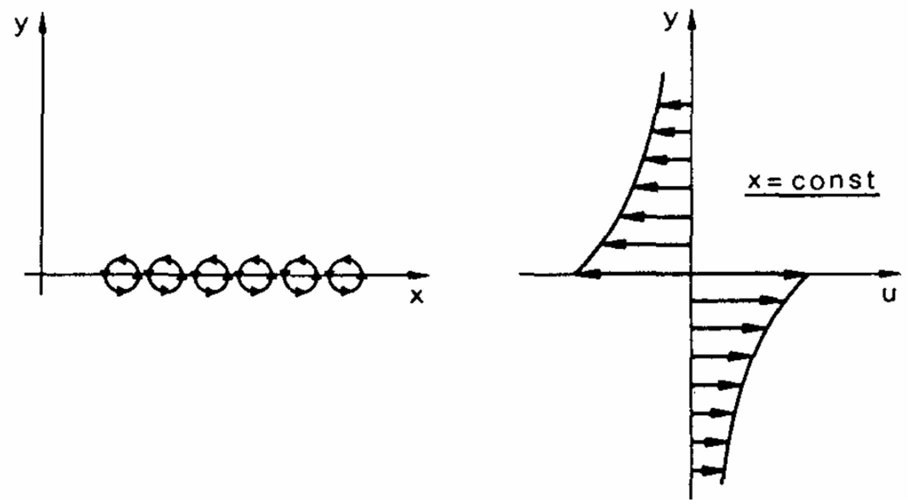

Figure 13. Vortex distribution on the *x*-axis. Left: arrangement of the vortices. Right: Profile of horizontal velocity above and below the vortex sheet. Taken from Schneider (1978).

If we exchange the expressions for φ and ψ in Eqs. (7.25) and (7.26), we obtain the following solution

$$\varphi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma(\xi) \tan^{-1} \frac{y}{x-\xi} \, \mathrm{d}\xi \, . \tag{7.31}$$

$$\psi = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma(\xi) \ln \sqrt{(x-\xi)^2 + y^2} \, \mathrm{d}\xi \, . \tag{7.32}$$

$$u = \frac{\partial \psi}{\partial y} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma(\xi) \frac{y}{(x-\xi)^2 + y^2} \, \mathrm{d}\xi \, , \tag{7.33}$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma(\xi) \frac{x-\xi}{(x-\xi)^2 + y^2} \, \mathrm{d}\xi \, . \tag{7.34}$$

which describes the flow near a vortex sheet, that is, a distribution of potential vortices of strength $\gamma(x)$ along the *x*-axis, as illustrated in Figure 13a. The two-dimensional potential vortices are infinitely extended in the third dimension, as illustrated in Figure 5.9 of Flandro et al. (2011). The profile of horizontal (tangential) velocity above and below the vortex sheet is shown in Figure 13b. The velocity is finite but discontinuous across the vortex sheet. Considering again a finite vortex sheet in the *x*-direction on the interval $x \in \langle 0, 1 \rangle$, we can take the limit of Eq. (7.33) for $y \to 0^{\pm}$ to obtain, in analogy to Eqs. (7.29) and (7.30),

$$u(x, 0^{\pm}) = \mp \frac{\gamma(x)}{2} , \tag{7.35}$$

$$\gamma(x) = u(x, 0^-) - u(x, 0^+) \, . \tag{7.36}$$

However, it is often necessary to determine $\gamma(x)$ from boundary conditions on v instead of u. Taking the limit of Eq. (7.34) for $y \to 0^{\pm}$ and $x \in \langle 0, 1 \rangle$ we obtain

$$v(x, 0) = \frac{1}{2\pi} \int_{0}^{1} \frac{\gamma(\xi)}{x-\xi} \, \mathrm{d}\xi \, , \tag{7.37}$$

where the Cauchy principal value of the integral is implied. Fortunately, there exists a formal solution of this integral equation for $\gamma(x)$,

$$\gamma(x) = \frac{1}{\sqrt{x(1-x)}} \left[C + \frac{2}{\pi} \int_{0}^{1} \frac{v(\xi, 0)}{\xi - x} \sqrt{\xi(1-\xi)} \, \mathrm{d}\xi \right] . \tag{7.38}$$

C is a free constant that can be determined by additional boundary conditions. C determines the total circulation Γ over any closed curve around the vortex sheet, which can be obtained by integration of the vortex distribution

$$\Gamma = \int_{0}^{1} \gamma(x) \, \mathrm{d}x \, . \tag{7.39}$$

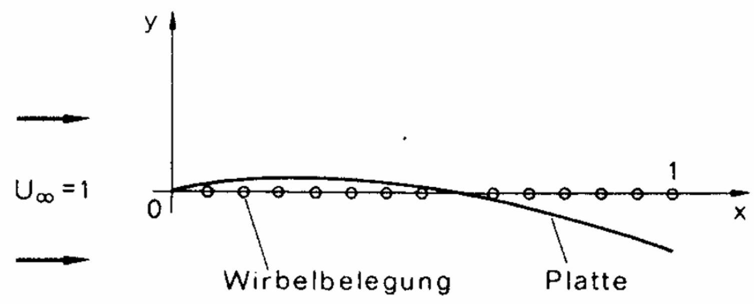

Figure 14: Curved and tilted plate, represented by a distribution of vortices, in a uniform incoming flow. Taken from Schneider (1978).

For example, the plane vortex sheet can be used to represent the flow over a curved and tilted plate (or slender airfoil), as illustrated in Figure 14, for small angles $\vartheta(x) \ll 1$ between the surface tangent of the plate/airfoil and the incoming flow. The asymptotic boundary condition for such flow reads

$$v(x,0) = \vartheta(x)\,. \tag{7.40}$$

The constant C (and thus also Γ) is then determined by the Kutta condition, which prescribes that the velocity is continuous at the trailing edge,

$$u(1,0^+) = u(1,0^-) \tag{7.41}$$

and thus,

$$\gamma(1) = u(1,0^-) - u(1,0^+) = 0\,. \tag{7.42}$$

Then,

$$C = \frac{2}{\pi}\int_0^1 \vartheta(x)\sqrt{\frac{x}{1-x}}\,\mathrm{d}x \tag{7.43}$$

and the solution for the vortex distribution reads

$$\gamma(x) = \frac{2}{\pi}\sqrt{\frac{1-x}{x}}\int_0^1 \frac{\vartheta(\xi)}{\xi - x}\sqrt{\frac{\xi}{1-\xi}}\,\mathrm{d}\xi\,. \tag{7.44}$$

7.5 Literature

Parts of this chapter are taken from the following sources:

- 'Singularitätenmethode', chapter 18 in Schneider (1978), pp. 114 – 126 and 139 - 145
- Techet (2005), pp. 6 - 12
- 'Potential Flows', chapter V. in Liburdy (2021)
- chapters 4.4 to 4.6 in Flandro et al. (2011), pp. 138 – 160