Similar solutions of the boundary-layer equations

See also chapter 7.2.1A of Schlichting & Gersten (2017)
Assume boundary layers with outer flow, $U(x) \neq 0$.

Similarity transformation

$$\xi = \bar{x}; \quad \eta = \frac{\bar{y}}{\delta(x)} \qquad (7.7) \qquad \rightarrow \frac{\partial}{\partial \bar{x}} = \frac{\partial}{\partial \xi} - \frac{\eta}{\bar{\delta}} \bar{\delta}' \frac{\partial}{\partial \eta}; \quad \frac{\partial}{\partial \bar{y}} = \frac{1}{\bar{\delta}} \frac{\partial}{\partial \eta}$$

⚠!! Applicable for $\bar{U}(\xi) \neq 0$

$$\bar{\Psi} = \bar{U}(\xi)\, \bar{\delta}(\xi)\, f(\eta)$$

(see Eq. 7.9 of S&G for general form)

$$\bar{u} = \frac{\partial \bar{\Psi}}{\partial \bar{y}} = \cancel{\frac{\partial \bar{\Psi}}{\partial \xi} \underbrace{\frac{\partial \xi}{\partial \bar{y}}}_{0}} + \frac{\partial \bar{\Psi}}{\partial \eta} \frac{\partial \eta}{\partial \bar{y}} = \bar{U}(\xi)\, \cancel{\bar{\delta}(\xi)}\, f'(\eta) \cdot \frac{1}{\cancel{\bar{\delta}(\xi)}}$$

$$\boxed{\frac{\bar{u}}{\bar{U}} = f'} \qquad (7.10)$$

$\Rightarrow$ BCs for f are independent of ξ!

$$-\bar{v} = \frac{\partial \bar{\Psi}}{\partial \bar{x}} = \frac{\partial \bar{\Psi}}{\partial \xi} \underbrace{\frac{\partial \xi}{\partial \bar{x}}}_{1} + \frac{\partial \bar{\Psi}}{\partial \eta} \frac{\partial \eta}{\partial \bar{x}} = \frac{d}{d\xi}(\bar{U}\bar{\delta}) f + \bar{U}\bar{\delta} f' \left(-\frac{\bar{y}}{\bar{\delta}^2} \bar{\delta}'\right)$$

$$= \frac{d}{d\xi}(\bar{U}\bar{\delta}) f - \bar{U} \eta \bar{\delta}' f' \qquad (7.11)$$

Substitute the similarity transformation into the b.-l. eq.:

$$\underbrace{\bar{U} f'}_{\frac{\partial \bar{\Psi}}{\partial \bar{y}}} \underbrace{\left(\frac{d\bar{U}}{d\xi} f' - \cancel{\frac{\eta}{\bar{\delta}} \frac{d\bar{\delta}}{d\xi} \bar{U} f''}\right)}_{\frac{\partial^2 \bar{\Psi}}{\partial \bar{x} \partial \bar{y}}} - \underbrace{\left[\frac{d(\bar{U}\bar{\delta})}{d\xi} f - \cancel{\bar{U} \eta \frac{d\bar{\delta}}{d\xi} f'}\right]}_{\frac{\partial \bar{\Psi}}{\partial \bar{x}}} \cdot \underbrace{\frac{\bar{U}}{\bar{\delta}} f''}_{\frac{\partial^2 \bar{\Psi}}{\partial \bar{y}^2}} = \bar{U} \frac{d\bar{U}}{d\xi} + \underbrace{\frac{\bar{U}}{\bar{\delta}^2} f'''}_{\frac{\partial^3 \bar{\Psi}}{\partial \bar{y}^3}} \quad \Big| \cdot \frac{\bar{\delta}^2}{\bar{U}}$$

Re-arrange the eq. for f to the following form:

$$f''' + \underbrace{\bar{\delta}\frac{d(\bar{U}\bar{\delta})}{d\xi}}_{\alpha_1} f f'' - \underbrace{\bar{\delta}^2 \frac{d\bar{U}}{d\xi}}_{\alpha_3} (f')^2 = \underbrace{-\bar{\delta}^2 \frac{d\bar{U}}{d\xi}}_{\alpha_2 = \alpha_3} \qquad (7.14)$$

$\Rightarrow$ Conditions for the existance of self-similar solutions:

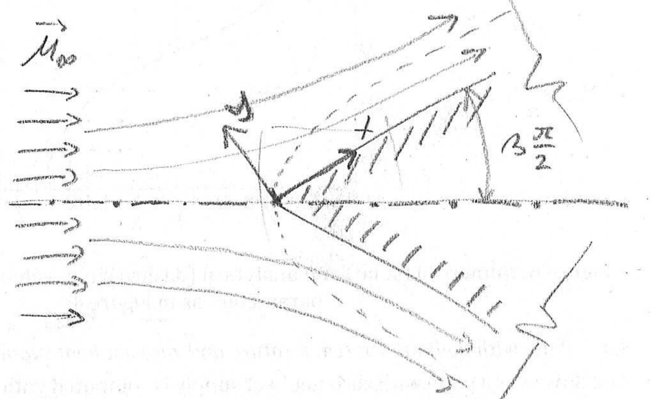
The solution f is independent of ξ if the coefficients $\alpha_{1,2,3}$ are independent of ξ:

$$\boxed{\begin{array}{c} \alpha_1 = \text{const.} \\ \alpha_2 = \alpha_3 = \text{const.} \end{array}}$$

Then, Eq. (7.14) describes a family of self-similar boundary-layer profiles, described by 2 parameters α_1 and $\alpha_2 = \alpha_3$.

Wedge flow and Falkner-Skan equation (S&G, p. 169)

For the purpose of the integral method, it is useful to consider the self-similar solution for $\alpha_1 > 0$. This case corresponds to a wedge flow:



See also Fig. 7.2b in S&G, p. 170

Without a loss of generality we can set

$$\alpha_1 = \bar{\delta}\frac{d}{d\xi}(\bar{U}\bar{\delta}) = 1, \quad (7.17)$$

because $\bar{\delta}$ is defined up to a free proportionality factor (see S&G, p. 169). With $\alpha_2 = \alpha_3 = \beta$ we obtain the Falkner-Skan equation:

$$\boxed{f''' + ff'' + \beta(1 - f'^2) = 0} \quad (7.15)$$

with the boundary conditions

$$\boxed{\begin{aligned} &f = 0,\ f' = 0 \quad \text{at } \eta = 0 \\ &f' = \frac{u}{U} = 1 \quad \text{at } \eta \to \infty \end{aligned}} \quad (7.16)$$

Thus, the b.l. profile depends on a single parameter β.

Remark:

The conditions for self-similar boundary layer read:

$$\alpha_1 = \bar{\delta}\frac{d}{d\xi}(\bar{U}\bar{\delta}) = 1 \quad , \quad \bar{\delta}^2 \frac{d\bar{U}}{d\xi} = \beta \qquad (7.17)$$

Thus, one can deduce the following form of $\bar{U}$ and $\bar{\delta}$:

$$\bar{U} = B\,\xi^m \quad ; \quad \bar{\delta} = \sqrt{\frac{2}{B(m+1)}}\;\xi^{\frac{1-m}{2}} \quad , \qquad (7.18)$$

where $m = \dfrac{\beta}{2-\beta}$. (7.19)

Thus, the outer velocity at the edge of the boundary layer is a power law. It turns out that the potential wedge flow (Fig. 7.26) satisfies this condition. For more details, see Schlichting & Gersten, pp. 169–170.