322.079 | 5.5.2025

Panel methods

Aim: Compute incompressible potential flow around a lifting airfoil of general aerodynamic shape (also thick profiles).

Method:

- Discretize the surface of the airfoil into (straight line) segments (panels).
- Distribute some singularities (vortex, source, dipole) along the surface of the airfoil
- Choose some simple parametrized distribution of the singularities along each panel (e.g. piecewise constant)

  $\overset{s.t.}{\Longrightarrow}$ solution described by N constants.
- Prescribe boundary conditions (flow tangency, Kutta cond.) at N control points

  $\Longrightarrow$ system of N algebraic equations for N unknown constants

 5.5.2025

Recall the problem formulation:

- Incompressible potential flow is governed by the Laplace equation:

$$\nabla^2 \phi = 0 \qquad (1)$$

- Boundary conditions:
  - i) far field: $\vec{u} \to \vec{u}_\infty$ as $x^2+y^2 \to \infty$ (2)
  - ii) flow tangency: $\vec{n} \cdot \vec{u} = 0$ at airfoil surface (3)
  - iii) Kutta condition: continuous & finite velocity & pressure at the trailing edge (4)

Panel method:

1) Expected form of the solution - superposition of analytical potential flows:

E.g.:

$$\underbrace{\phi_\infty}_{\text{far field}} \qquad \overbrace{\phi_s}^{\text{source distribution along the surface}} \qquad \underbrace{\phi_v}_{\text{vortex distribution along the surface}}$$

$$\phi = \phi_\infty + \phi_s + \phi_v \qquad (5)$$

$\nabla^2 \phi_\infty = 0$, $\nabla^2 \phi_s = 0$, $\nabla^2 \phi_v = 0$ such that $\nabla^2 \phi = 0$

a) Φ_∞ satisfies the far-field condition:

$$\boxed{\Phi_\infty = x \cdot \cos\alpha + y \cdot \sin\alpha} \qquad (6)$$

$\Rightarrow$ BCs for Φ_s and Φ_v:

i) <u>Far field:</u>

$$\Phi_s + \Phi_v \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty \qquad (7)$$

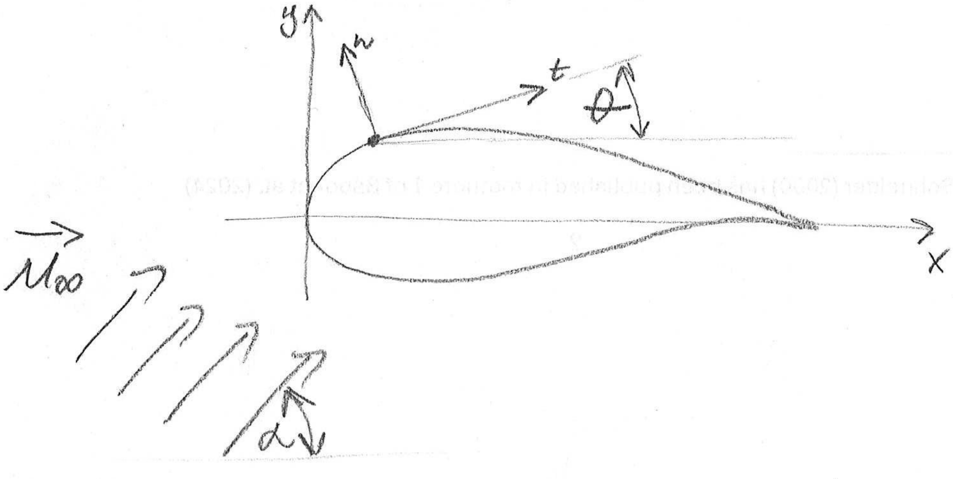
ii) <u>Flow tangency:</u>

$$u_n \equiv \vec{n} \cdot \vec{u} \equiv \vec{n} \cdot \vec{\nabla} \Phi \equiv \frac{\partial \Phi_\infty}{\partial n} + \frac{\partial \Phi_s}{\partial n} + \frac{\partial \Phi_v}{\partial n} \overset{!}{=} 0 \quad \text{at surface}$$

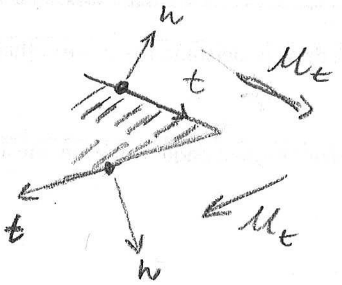
$$\frac{\partial \Phi_s}{\partial n} + \frac{\partial \Phi_v}{\partial n} \overset{!}{=} - \frac{\partial \Phi_\infty}{\partial n}$$

$$\frac{\partial \Phi_s}{\partial n} + \frac{\partial \Phi_v}{\partial n} \overset{!}{=} \sin(\theta - \alpha) \quad \text{at surface} \qquad (8)$$

where θ is the local inclination of the surface.



iii) Kutta condition



$$-u_t\Big|_{\substack{\text{lower}\\\text{surface}}} = u_t\Big|_{\substack{\text{upper}\\\text{surface}}} \quad \text{at trailing edge} \tag{9}$$

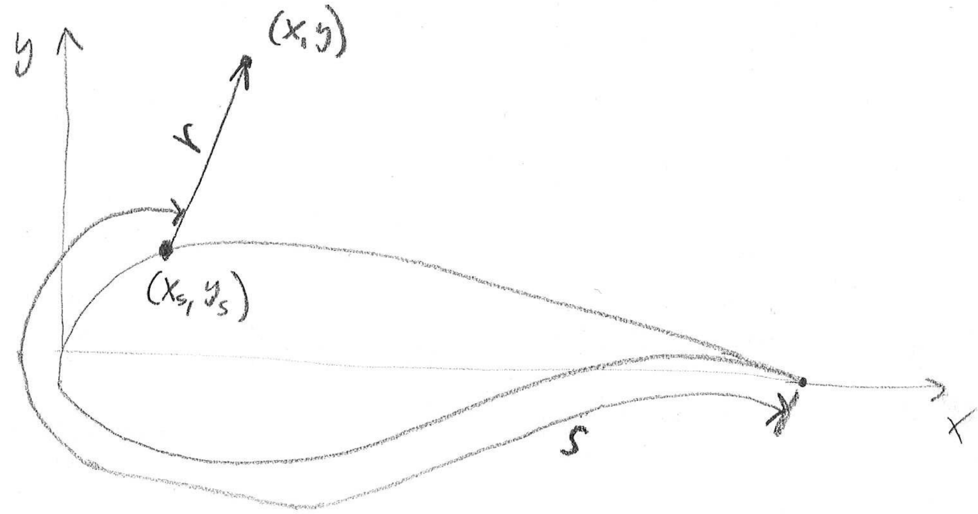
where $u_t = \vec{t} \cdot \vec{u}$

b) ϕ_s as distribution of sources along the surface

$$\phi_s(x,y) = \frac{1}{2\pi} \oint q\big(x_s(s), y_s(x)\big) \ln \underbrace{\sqrt{(x-x_s(s))^2 + (y-y_s(s))^2}}_{r} \, ds \tag{10}$$

airfoil surface

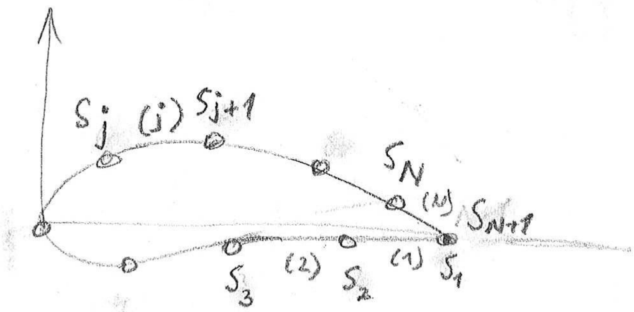
arc length



c) ϕ_v - distribution of vortices along the surface

$$\phi_v(x,y) = -\frac{1}{2\pi} \oint_{\substack{\text{airfoil} \\ \text{surface}}} \mu(x_s(s), y_s(s)) \arctan \frac{y - y_s(s)}{x - x_s(s)} \, ds \qquad (11)$$

2) Discretize the surface into N+1 "nodes" connected by N straight "panels":



a) Select some simple parametrized form of $q(s)$ and $\mu(s)$.
E.g., piecewise constant:

$$q(s) = q_j \quad , \; j = 1, \dots, N$$

$$\mu(s) = \mu = \text{const.}$$

(Hesse-Smith method)

Different panel methods use different forms of $q(s)$ & $\mu(s)$

b) $$\oint \approx \sum_{j=1}^{N} \int_{s_j}^{s_{j+1}} \Rightarrow \phi_s(x,y) \approx \frac{1}{2\pi} \sum_{j=1}^{N} q_j \int_{s_j}^{s_{j+1}} \ln r \, ds \qquad (12)$$

$$\phi_v(x,y) \approx -\frac{\mu}{2\pi} \sum_{j=1}^{N} \int_{s_j}^{s_{j+1}} \arctan \frac{y - y_s}{x - x_s} \, ds \qquad (13)$$

322.079 | Velocity: | 5.5.2025 |

$$u_s(x,y) = \frac{\partial \phi_s}{\partial x} = \frac{1}{2\pi} \sum_{j=1}^{N} q_j \int_{S_j}^{S_{j+1}} \frac{x - x_s(s)}{(x - x_s(s))^2 + (y - y_s(s))^2} \, ds$$

$$v_s(x,y) = \frac{\partial \phi_s}{\partial y} = \frac{1}{2\pi} \sum_{j=1}^{N} q_j \int_{S_j}^{S_{j+1}} \frac{y - y_s(s)}{(x - x_s(s))^2 + (y - y_s(s))^2} \, ds$$

$$u_v(x,y) = \frac{\partial \phi_v}{\partial x} = -\frac{\gamma}{2\pi} \sum_{j=1}^{N} \int_{S_j}^{S_{j+1}} \frac{y - y_s}{(x - x_s)^2 + (y - y_s)^2} \, ds$$

$$v_v(x,y) = \frac{\partial \phi_v}{\partial y} = \frac{\gamma}{2\pi} \sum_{j=1}^{N} \int_{S_j}^{S_{j+1}} \frac{x - x_s}{(x - x_s)^2 + (y - y_s)^2} \, ds$$

$$u = u_\infty + u_s + u_v \, , \quad u_\infty = \cos\alpha$$

$$v = v_\infty + v_s + v_v \, , \quad v_\infty = \sin\alpha$$

322.079 | 5.5.2025

$\phi = \phi_\infty + \phi_s + \phi_v$ determined by N+1 parameters: $q_{1,\dots,N}$ and γ

3) Discretize flow tangency BC:

- enforce eq. (8) at N discrete control points $(\bar{x}_i, \bar{y}_i)$ at the centers of the panels:

$$(\bar{x}_i, \bar{y}_i) = \left(\frac{x_i + x_{i+1}}{2}, \frac{y_i + y_{i+1}}{2}\right) \quad (14)$$

$$\boxed{\vec{n} \cdot \vec{u}\,(\bar{x}_i, \bar{y}_i) \overset{!}{=} 0}, \quad i = 1, \dots, N \quad (15)$$

(Kutta condition provides the N+1 st eq.)

$$\vec{n}_i = (-\sin\theta_i, \cos\theta_i) \quad ; \quad \vec{u} = \vec{u}_\infty + \vec{u}_s + \vec{u}_v$$

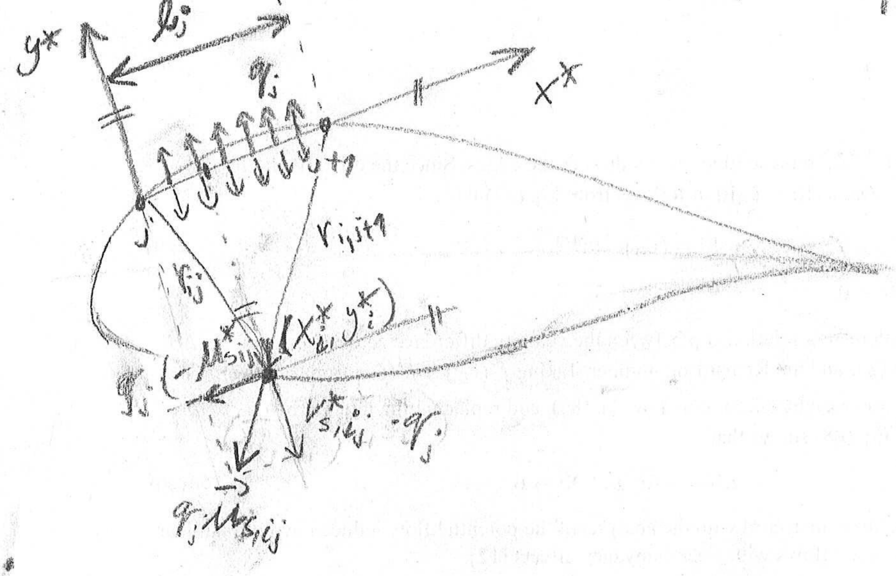
$$u_{n,i} \equiv \vec{n}_i \cdot \vec{u}_i = -u_i \sin\theta_i + v_i \cos\theta_i$$

$$\vec{u}_i \equiv \vec{u}(\bar{x}_i, \bar{y}_i) \cancel{= \vec{u}_\infty(\bar{x}_i, \bar{y}_i) + \vec{u}_s(\bar{x}_i, \bar{y}_i) + \vec{u}_v(\bar{x}_i, \bar{y}_i)}$$

322.079

4) Analytical integrals [5.5.2025]

$\vec{n}\cdot\vec{u}$ and $\vec{t}\cdot\vec{u}$ can be expressed analytically in a local coordinate system, oriented with the j-th panel

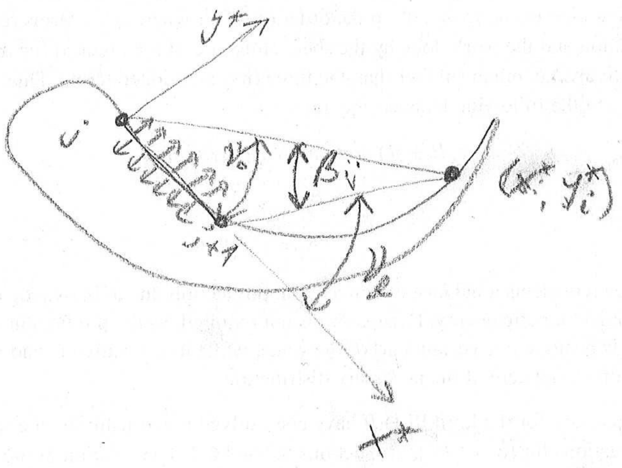


$$u^*_{sij} = \frac{1}{2\pi}\int_0^{l_j} \frac{x_i^* - t}{(x_i^* - t)^2 + y^{*2}}\,dt = -\frac{1}{2\pi}\ln\frac{r_{i\,j+1}}{r_{ij}}$$

Recall: For vortex distribution:

$$u^*_{vij} = V^*_{sij}$$

$$V^*_{vij} = -u^*_{sij}$$



$$V^*_{sij} = \frac{1}{2\pi}\int_0^{l_j} \frac{y_i^*}{(x_i^* - t)^2 + y^{*2}}\,dt$$

$$= \frac{1}{2\pi}\left[\arctan\frac{y^*}{x^* - t}\right]_{t=0}^{t=l_j}$$

$$= \frac{\beta_{ij}}{2\pi}$$

$$\nu_l - \nu_0 = \beta_{ij}$$

can be re-arranged with trigon. identity to compute arctan only once :)

See Moran, p. 109, eq. (4-87)

3.5.2025

Subscripts:

i … control point

j … panel with singularity distribution

Flow tangency condition:

$$\vec{n}_i \cdot \vec{u} \stackrel{!}{=} 0 \quad \text{at} \quad (\overline{x}_i, \overline{y}_i)$$

$$\vec{n}_i = (-\sin\theta_i, \cos\theta_i)$$

$$\vec{u} = u\,\vec{e}_x + v\,\vec{e}_y = u_j^*\,\vec{e}_{x_j^*} + v_j^*\,\vec{e}_{y_j^*}$$

$$\vec{e}_{x_j^*} = (\cos\theta_j, \sin\theta_j) \; ; \; \vec{e}_{y_j^*} = (-\sin\theta_j, \cos\theta_j)$$

$$\underline{\underline{\vec{n}_i \cdot \vec{u}_{ij}}} = u_{ij}^* \cdot (\vec{n}_i \cdot \vec{e}_{x_j^*}) + v_{ij}^* (\vec{n}_i \cdot \vec{e}_{y_j^*})$$

$$= u_{ij}^* \cdot (-\sin\theta_i \cdot \cos\theta_j + \cos\theta_i \cdot \sin\theta_j)$$

$$+ v_{ij}^* \cdot (+\sin\theta_i \sin\theta_j + \cos\theta_i \cdot \cos\theta_j)$$

$$\underline{\underline{= -u_{ij}^* \cdot \sin(\theta_i - \theta_j) + v_{ij}^* \cdot \cos(\theta_i - \theta_j)}}$$

$$\vec{u} = \vec{u}_\infty + \vec{u}_s + \vec{u}_v$$

$$\vec{u}_i = \vec{u}(\overline{x}_i, \overline{y}_i) = \vec{u}_\infty + \sum_{j=1}^{N} q_j\,\vec{u}_{s_{ij}} + \gamma \sum_{j=1}^{N} \vec{u}_{v_{ij}}$$

$$\vec{u}_\infty = \cos\alpha\,\vec{e}_x + \sin\alpha\,\vec{e}_y$$

$$\vec{u}_{s_{ij}} = u_{s_{ij}}^*\,\vec{e}_{x_j^*} + v_{s_{ij}}^*\,\vec{e}_{y_j^*}$$

$$\vec{u}_{v_{ij}} = u_{v_{ij}}^*\,\vec{e}_{x_j^*} + v_{v_{ij}}^*\,\vec{e}_{y_j^*}$$

$$u_{v_{ij}}^* = v_{s_{ij}}^*$$

$$v_{v_{ij}}^* = -u_{s_{ij}}^*$$

5.5.2025

$$\vec{n}_i \cdot \vec{u}_i = \underbrace{\vec{n}_i \cdot \vec{u}_\infty + \sum_{j=1}^{N} q_j \vec{u}_{s_{ij}} \vec{n}_i}_{} + \gamma \sum_{j=1}^{N} \vec{n}_i \cdot \vec{u}_{v_{ij}} \stackrel{!}{=} 0 \quad \text{for } \forall\, i = 1, \dots N$$

$$\vec{n}_i \cdot \vec{u}_\infty = -\sin\theta_i \cdot \cos\alpha + \cos\theta_i \cdot \sin\alpha = -\sin(\theta_i - \alpha)$$

$$\vec{u}_{s_{ij}} \cdot \vec{n}_i = \frac{1}{2\pi} \ln \frac{r_{ij+1}}{r_{ij}} \cdot \sin(\theta_i - \theta_j) + \frac{\beta_{ij}}{2\pi} \cos(\theta_i - \theta_j)$$

$$\vec{u}_{v_{ij}} \cdot \vec{n}_i = \frac{\beta_{ij}}{2\pi} \cdot (-1) \sin(\theta_i - \theta_j) + \frac{1}{2\pi} \ln \frac{r_{ij+1}}{r_{ij}} \cdot \cos(\theta_i - \theta_j)$$

Linear system of (N) algebraic equations with N+1 unknowns:
$\Rightarrow$ requires the Kutta condition:

$$\vec{t}_1 \cdot \vec{u}_1 + \vec{t}_N \cdot \vec{u}_N \stackrel{!}{=} 0$$

$$\vec{t}_i = (\cos\theta_i \,;\, \sin\theta_i) = \vec{e}_x \cos\theta_i + \vec{e}_y \sin\theta_i$$

$$\vec{t}_i \cdot \vec{u}_i = \vec{t}_i \cdot \vec{u}_\infty + \sum_{j=1}^{N} q_j \vec{u}_{s_{ij}} \cdot \vec{t}_i + \gamma \sum_{j=1}^{N} \vec{u}_{v_{ij}} \cdot \vec{t}_i$$

$$\vec{t}_i \cdot \vec{u}_\infty = \cos\theta_i \cos\alpha + \sin\theta_i \sin\alpha = \cos(\theta_i - \alpha)$$

$$\vec{u}_{s_{ij}} \cdot \vec{t}_i = u^*_{s_{ij}} \cdot (\cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j) + v^*_{s_{ij}} (-\cos\theta_i \sin\theta_j + \sin\theta_i \cos\theta_j)$$

$$= u^*_{s_{ij}} \cdot \cos(\theta_i - \theta_j) + v^*_{s_{ij}} \cdot \sin(\theta_i - \theta_j)$$

$$= -\frac{1}{2\pi} \ln \frac{r_{ij+1}}{r_{ij}} \cos(\theta_i - \theta_j) + \frac{\beta_{ij}}{2\pi} \sin(\theta_i - \theta_j)$$

$$\vec{u}_{v_{ij}} \cdot \vec{t}_i = u^*_{v_{ij}} \cos(\theta_i - \theta_j) + v^*_{v_{ij}} \sin(\theta_i - \theta_j)$$

$$= \frac{\beta_{ij}}{2\pi} \cos(\theta_i - \theta_j) + \frac{1}{2\pi} \ln \frac{r_{ij+1}}{r_{ij}} \sin(\theta_i - \theta_j)$$