322/079 23.5.2025

Integral momentum equation

Section 7.6.1. of Schlichting & Gersten (2017)

- Start from the x-momentum equation in dimensional form:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{dU}{dx} = \frac{1}{\rho}\frac{\partial}{\partial y}\Big(\underbrace{\mu\frac{\partial u}{\partial y}}_{\tau}\Big) \qquad (7.4)$$

$$\int_0^h \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{dU}{dx}\right)dy = \frac{1}{\rho}\left[\mu\frac{\partial u}{\partial y}\right]_{y=0}^{y=h}, \quad h > \delta \;\; \forall x$$

$$= -\frac{1}{\rho}\left(\mu\frac{\partial u}{\partial y}\right)_{y=0} = -\frac{\tau_w}{\rho}$$

- Continuity equation:

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \qquad (7.5)$$

substitute

$$\boxed{v = -\int_0^y \frac{\partial u}{\partial x}\,dy}$$

$$\Rightarrow \int_0^h \Big(u\frac{\partial u}{\partial x} - \underbrace{\frac{\partial u}{\partial y}\int_0^y \frac{\partial u}{\partial x}\,dy} - U\frac{dU}{dx}\Big)dy = -\frac{\tau_w}{\rho}$$

The second term on the LHS can be integrated by parts:

$$\int_0^h \left(\frac{\partial u}{\partial y}\int_0^y \frac{\partial u}{\partial x}\,dy\right)dy = \left[u\int_0^y \frac{\partial u}{\partial x}\,dy\right]_{y=0}^{y=h} - \int_0^h u\frac{\partial u}{\partial x}\,dy$$

$$= U\int_0^h \frac{\partial u}{\partial x}\,dy - \int_0^h u\frac{\partial u}{\partial x}\,dy$$

Thus, the momentum-integral equation can be written as:

$$\int_0^h \left(2u\frac{\partial u}{\partial x} - U\frac{\partial u}{\partial x} - U\frac{dU}{dx}\right)dy = -\frac{\tau_w}{\rho}$$

$$-\int_0^h \left[\frac{\partial u}{\partial x}(U-u) + u\left(\frac{dU}{dx} - \frac{\partial u}{\partial x}\right) - u\frac{dU}{dx} + U\frac{dU}{dx}\right]dy = -\frac{\tau_w}{\rho}$$

$$\int_0^h \frac{\partial}{\partial x}\left[u(U-u)\right]dy \qquad + \frac{dU}{dx}\int_0^h (U-u)\,dy = \frac{\tau_w}{\rho}$$

- Note that, in this form, both integrands vanish outside the boundary layer at $y = h > \delta$. Thus, the upper bounds h can be replaced by ∞.
- Also, the differentiation $\frac{\partial}{\partial x}$ in the first integrand can be moved in front of the integral:

$$\frac{d}{dx}\underbrace{\int_0^\infty u(U-u)\,dy}_{\delta_2 U^2} + \frac{dU}{dx}\underbrace{\int_0^\infty (U-u)\,dy}_{\delta_1 U} = \frac{\tau_w}{\rho}$$

Next, we introduce displacement thickness δ_1 and momentum thickness δ_2, such that:

$$\delta_1 U = \int_0^\infty (U-u)\,dy \quad (7.98) \qquad \text{and} \qquad \delta_2 U^2 = \int_0^\infty u(U-u)\,dy \,. \quad (7.99)$$

Note that δ_1 and δ_2 are properties of a given boundary-layer profile.

The variation of δ_1 and δ_2 along the boundary layer must satisfy the momentum-integral equation:

$$\boxed{\frac{d}{dx}\left(\delta_2 U^2\right) + \delta_1 U \frac{dU}{dx} = \frac{\tau_w}{\rho}} \qquad (7.100)$$

Note that Eq. (7.100) is a single ODE with 2 unknowns δ_1 and δ_2. To close the problem, we can assume a certain form of the boundary-layer velocity profile (model). For example, assuming the single-parameter Hartree profiles, governed by the Falkner-Skan equation, we can find the relationship between δ_1 and δ_2 and close the problem.