Integral method

for computing the boundary layer

- Pre-CFD method
- Approximate solution of the B.L. equations

General procedure

1) Select some simple and reasonable form of the velocity profile.
   a) parametrized analytical function (spline, ...)
   b) Self-similar b.L. solution (also parametrized)

2) Find/compute the relationships between the parameters describing the profile (selected in step 1) and the quantities of interest (τ_w, δ, ...)

3) Integrate the conservation laws (momentum, energy, ...) over the b.L. thickness, $\int_0^{\delta} \dots dy$
   $\Rightarrow$ integral conservation laws
   $\rightarrow$ substitute the assumed parametric form of the profile
   $\rightarrow$ obtain ODE(s) for the variation of the parameters along the surface.

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4) Solve the integral equations for the variation of the shape factors along the boundary layer.

→ Using the parametrized b.l. profiles obtained in 2), obtain the variation of τ_w and δ along the boundary layer.

start from the stagnation point as the initial point.

Transition to turbulent boundary layer

- Once the boundary layer becomes turbulent, we must extend the b.l. eqs with a turbulence model (to model the effect of the vortices)
  - → for the simple template problem we obtain different simplified equations with possibly different shape factors

1) Complete the laminar boundary layer, starting from the stagnation point

2) Estimate the location of transition to turbulence

3) Compute the turbulent b.l., starting from the transition point. Use the laminar b.l. as an initial condition.

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Integral method for laminar boundary layer

Recap:

1) Use analytical velocity profile obtained from simple geometry 

2) Describe velocity profile by small number of parameters

3) Find expressions for τ_w, δ_2 depending on the parameters

4) Compute the variation of the parameters along the surface

   * Test if laminar separation occurs

5) Compute τ_w, δ_2 → Friction drag force

6) Correct the potential flow using δ_2

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Outlook – Transition to turbulence

- Turbulent boundary layer
  - estimate transition point (e.g., pressure minimum)
  - Laminar boundary layer from stagnation to transition points
  - Turbulent boundary layer from transition point to trailing edge
- Test separation criteria for turbulent region

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Solution of B.L. eqs. with integral methods: overview

0) • Motivated by analytical solution for specific shapes (plane wall)

1) Transformation of B.L. eqs.

2) Assume specific form of the velocity profile, defined by small number of parameters (depending on x)

0) Provides exact results for simple geometries

3) Determine variation of the parameters along the surface $\vec{F} = m\vec{a}$
   - described by integral equations < Global conservation of momentum / Global conservation of energy
   - integral of some expression (known from the outer potential flow) depending on the tangential velocity from the outer flow

4) • Compute the velocity profiles for the given values of the parameters
   - described by ODE for a scaled stream function $f(\eta)$ in stretched coordinates η
   - may be implicit

5) • Compute skin friction and displacement thickness from the velocity profile

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Hartree profiles: Actual computational procedure:

- solve Falkner-Skan for a range of $\beta \in (-0.199, 0.999)$ and tabulate important quantities

- Compute $Z(s)$, $\Gamma(s)$

- obtain wall shear stress & displacement thickness from tabulated data

Recapitulation: Integral methods for boundary layers (profile fitting)

- Boundary layer eqs. (neglected wall curvature, $R \gg \delta$)
- integrate momentum eq. over δ
  - $\Rightarrow$ integral momentum equation (von Karman's) (Moran, p. 204 or S&G, p. 192)
    - reduced PDE to ODE
    - valid for both, turbulent AND laminar B.L.
    - too many unknowns $\Rightarrow$ closure

  - analogously, one can ~~define~~ find integral energy eq. etc.

- assume certain shape of $u(y)$, depending on shape factors
  - $\Rightarrow$ dependence of c_f, δ_1, ... on shape factors
- substitute expected profile into integral eq(s).
  - $\Rightarrow$ ODE(s) for shape factors vs. s

Laminar B.L. – Hartree profiles (S.&G., pp. 196–201)

- shape factors Z, $\Gamma(Z)$
  - $\rightarrow$ ODE integrated formally: $Z(x) = f(U_e) \cdot \int_0^x g(U_e)\,dx$
    - integral computed numerically
- dependence of c_f, δ_1 on Γ obtained from (tabulated) solutions of Falkner–Skan eqs., for a range of β

p. 169 of S&G, $\rightarrow$ tabulated on p. 197

Turbulent boundary layer: computational workflow

- Start with laminar boundary layer:
  1) Find stagnation point: $U_e(s_0) = 0$
  2) Compute laminar B.L. from s_0 until s_t
     - criterion for s_t: e.g. $\left.\frac{dP_e}{ds}\right|_{s=s_t} = 0$
  3) Use laminar solution at s_t as initial condition for turbulent B.L.
  4) Compute turbulent B.L. from s_t until trailing edge or separation

Higher-order boundary-layer theory :

- displacement thickness δ_1 of the B.L. modifies the outer potential flow

- weak coupling (higher order B.L.T.) :

  1) Compute outer flow without B.L.

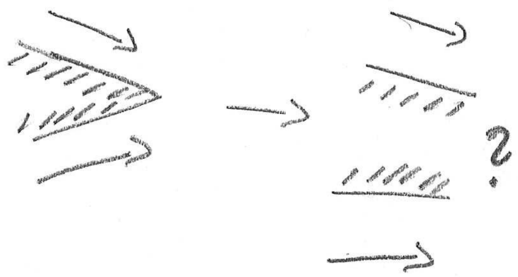
  2) Compute B.L. $\rightarrow \delta_1$

  3) Correct outer flow using δ_1

     a) modify body shape by δ_1

     b) prescribe normal displacement velocity at wall : $U_n\Big|_{\frac{y}{L}\to 0} = \frac{d}{dx}(U_e \delta_1)$

⚠ Trailing edge :



- displacement thickness of the wake
  $\Rightarrow$ no stagnation pressure

- Singularity $\Rightarrow$ strong interaction
  - outer flow & B.L. solved simultaneously

  $\Rightarrow$ Tripple-Deck theory $\Rightarrow$ c_D, c_f correction