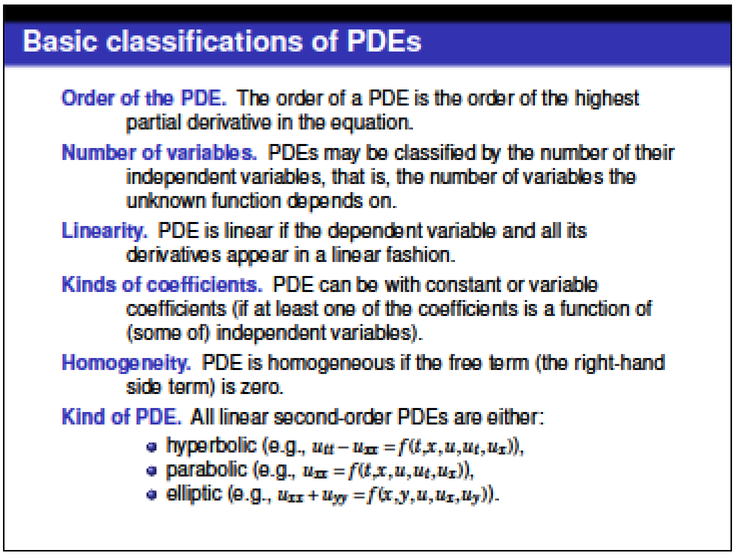
3 Types of partial differential equations

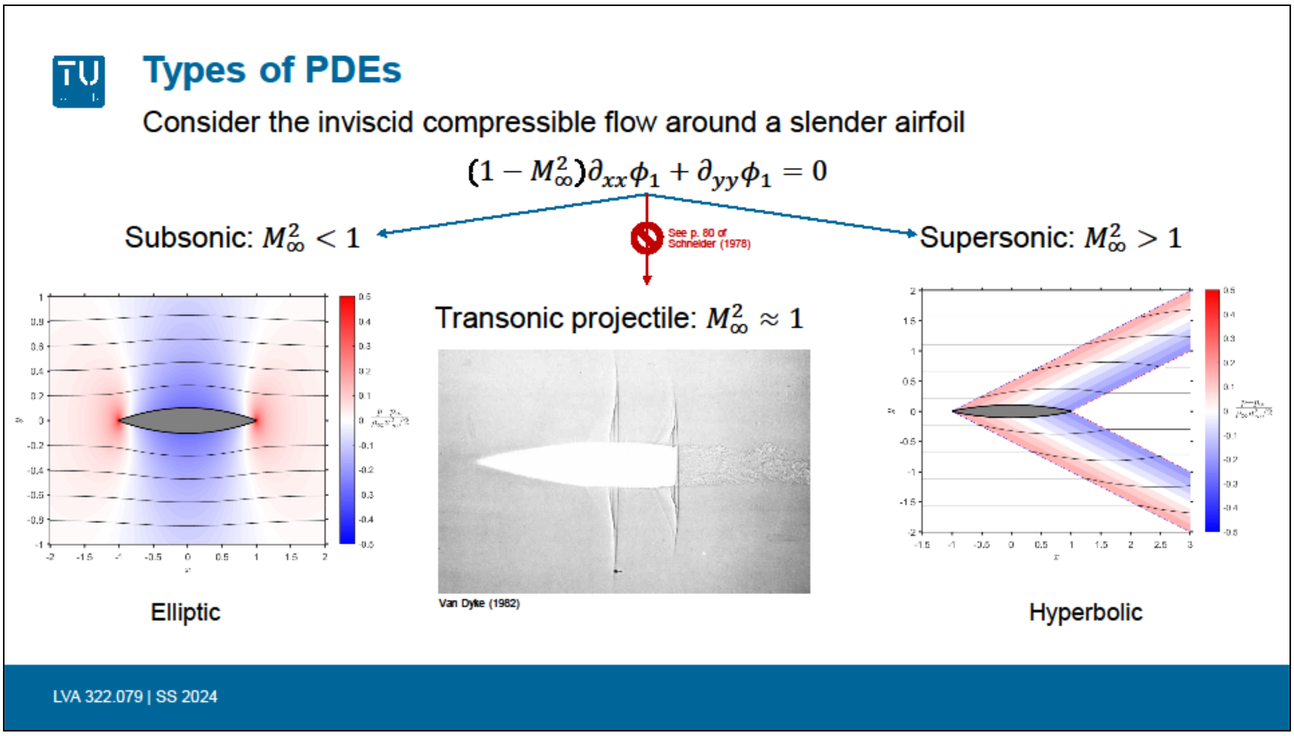


Some PDEs and systems of PDEs can be classified as either elliptic, parabolic or hyperbolic. For example, all linear second-order PDEs fall into one of these types. The type of PDE determines certain properties of its solution and imposes restrictions on boundary conditions and discretization methods which can be used to solve it numerically.

3.1 Properties of different types of PDEs and their solutions

- Elliptic
  - Boundary-value problems. Boundary conditions must be prescribed at all boundaries.
  - Local change of the solution instantly influences the solution everywhere else.
  - No propagation of waves, no propagation of discontinuities (like shock waves).
  - The solution tends to be smooth (except, possibly, at the boundaries). Discontinuities of the solution are associated with singularities.
- Hyperbolic
  - Waves (or discontinuities like shock waves) can propagate in different directions.
  - Waves can reflect.
  - Local boundary values affect the solution only inside a certain region of influence, bounded by “characteristics”.
- Parabolic
  - Parabolic can be viewed as an intermediate type.
  - Waves can propagate only in one specific direction and cannot reflect. The waves propagate in the direction of a time-like independent variable (information most likely cannot travel backwards in time, at least in the context of classical mechanics).
  - In other directions, information propagates instantly.
  - Parabolic equations often describe time-evolution of a solution, which has elliptic properties when frozen in time.

3.2 Examples of different types of PDEs



As an example of different types of PDEs, we consider the equation that describes the leading-order perturbation of a uniform flow of compressible fluid by a thin profile. The equation is given in the slide above. The x-axis is aligned with the incoming velocity. The dependent variable ϕ_1 is the potential of the velocity perturbation, defined such that the horizontal and vertical velocity perturbation components u_1 and v_1, respectively, are the x- and y-derivatives of ϕ_1:

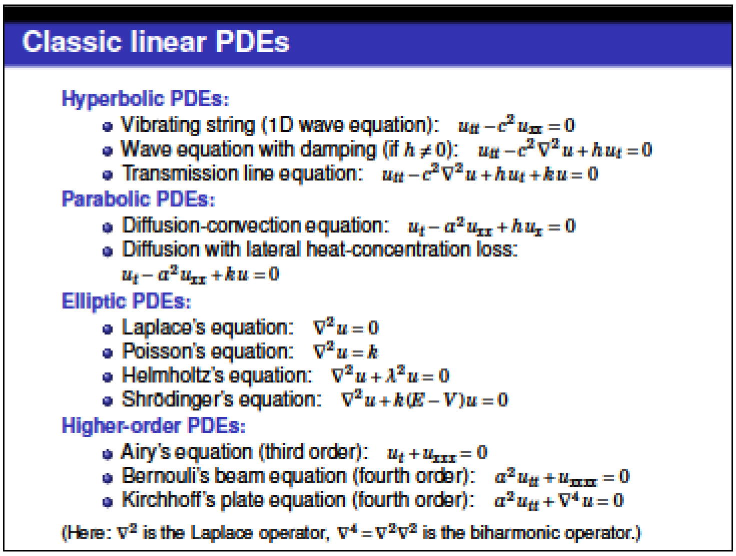
$$u_1 = \partial\phi_1/\partial x, \qquad v_1 = \partial\phi_1/\partial y. \tag{3.1}$$

The parameter M_∞ is the far-field Mach number, which is the ratio of the incoming velocity u_∞ and the far-field speed of sound c_∞. The governing equation for ϕ_1, given in the slide, is valid only when M_∞ is sufficiently smaller or larger than 1. It is not applicable to a transonic flow when $M_\infty \approx 1$, see p. 80 of Schneider (1978) for further details.

For $M_\infty < 1$, the equation is elliptic. Then, it describes a subsonic flow, which has typical properties of a solution of an elliptic equation. The streamlines and pressure field of the outer flow are shown on the left side of the slide above. The streamlines are smooth, and the pressure field is continuous. The profile modifies the flow everywhere around it (in all directions). The influence of the profile in the flow decays asymptotically with increasing distance.

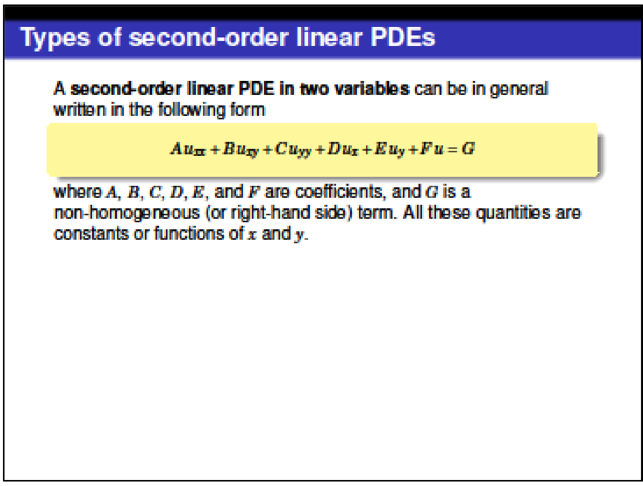
For $M_\infty > 1$, the equation is hyperbolic and describes a supersonic flow. The solution for the supersonic flow is shown on the right side of the slide. Notice that the profile affects the flow only inside a certain region of influence. Namely, a shock wave is initiated at the leading edge. The shock wave spreads from the profile at certain angle, depending on M_∞, with respect to the direction of the incoming flow. Upstream of the shock wave, the flow is not affected by the presence of the profile. At the shock wave front, the pressure is discontinuous, as well as the direction of the streamlines. There is also a second shock wave originating at the trailing edge. The waves could also reflect from other objects.

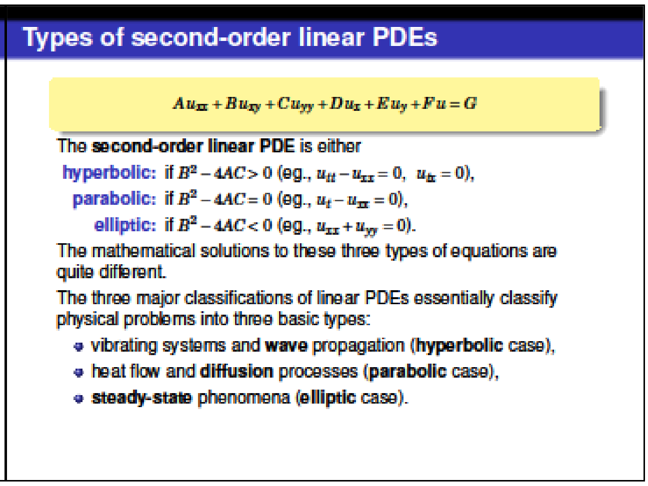
For $M_\infty = 1$, the equation would be parabolic. Although it is not valid for a real transonic flow, we may still study its properties. Note that the first term drops out, such that the equation describes a solution where ϕ_1 varies linearly in y-direction and v is constant in y-direction. It can be shown (we will see how) that any wave fronts described by the equation for $M_\infty = 1$ are vertical. Although the equation is not valid for the flow considered here, the wave fronts perpendicular to the incoming flow can be observed in experiments at transonic conditions, as seen in the middle picture of the slide above. The photograph is taken from Van Dyke (1982).



This slide provides some further examples of PDEs of each type.

3.3 How to determine the type of a second-order linear PDE





The type of a linear second-order PDE with two independent variables (x and y in the slides above) is determined by the coefficients *A, B, C,* multiplying the highest (second) derivatives of the solution, where *B* multiplies the cross-derivative. The criterion for determining the type from these coefficients is provided in the right slide.

In the case of more than two independent variables, x_i, the second-order PDE can be written in general form as follows:

$$a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i \frac{\partial u}{\partial x_i} + cu = f. \tag{3.2}$$

The PDE is:

- Elliptic if the coefficient matrix $a_{i,j}$ is either positive or negative definite, that is, all its eigenvalues are nonzero and have the same sign;
- Hyperbolic if all eigenvalues of $a_{i,j}$ are non-zero, and all but one have the same sign;
- Parabolic if the matrix $a_{i,j}$ is semi-definite, that is, when exactly one eigenvalue is zero while all others have the same sign, and, furthermore, $\mathrm{rank}(a_{i,j} | \, b_i)$ equals the number of independent variables.

3.4 How to determine the type of a system of first-order linear PDEs

The system of n first-order linear PDEs with two independent variables can be written in general form as follows:

$$A_{i,j}\frac{\partial u_j}{\partial x} + B_{i,j}\frac{\partial u_j}{\partial y} + C_{i,j}u_j = f_i. \tag{3.3}$$

The type of the system depends on the number of solutions of the following equation:

$$\det(\boldsymbol{A}^{\mathrm{T}}\mathrm{d}y - \boldsymbol{B}^{\mathrm{T}}\mathrm{d}x) = 0 \tag{3.4}$$

for either $\mathrm{d}y/\mathrm{d}x$ or $\mathrm{d}x/\mathrm{d}y$. The system is:

- elliptic if there are no real solutions,
- hyperbolic if there are n real solutions,
- parabolic if there are between 1 and $n-1$ real solutions and no complex solutions.

If there are both real and complex solutions, then the type of the system remains undetermined.

The procedure is similar when there are more than two independent variables. In that case, there are simply more coefficient matrices, and the system can be written analogously as

$$A_{i,j}\frac{\partial u_j}{\partial x_1} + B_{i,j}\frac{\partial u_j}{\partial x_2} + C_{i,j}\frac{\partial u_j}{\partial x_3} + \cdots = f_i. \tag{3.5}$$

The type of the system is then determined by the number of solutions of the equation

$$\det(\boldsymbol{A}^{\mathrm{T}}\lambda_1 + \boldsymbol{B}^{\mathrm{T}}\lambda_2 + \boldsymbol{C}^{\mathrm{T}}\lambda_3 + \cdots) = 0 \tag{3.6}$$

for, e.g., $\lambda_1 = g(\lambda_2, \lambda_3, \ldots)$. The criterion is the same as for two independent variables.

3.5 Characteristics

A characteristic of a (system of) PDE(s) with two independent variables is a certain set of curves in the space of the independent variables. For example, the simplest form of a characteristic in two-dimensional Cartesian coordinates (x, y) is

$$y(x) = y_0 + \frac{\mathrm{d}y}{\mathrm{d}x}x, \tag{3.7}$$

where y_0 parametrizes the curves and $\mathrm{d}y/\mathrm{d}x$ is their slope. The set of curves is a characteristic if the variation of the solution along the curves can be described by ODE(s) instead of PDE(s). The variation of the solution of the PDE(s), say $u(x, y)$, along a curve $y(x)$ can be written as $u(x, y(x))$. When $y(x)$ is a characteristic, then all partial derivatives in the PDE(s) can be replaced with (a linear combination of) total derivatives along $y(x)$. For example, the first total derivative of $u(x, y(x))$ can be written as

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\partial u}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\partial u}{\partial y}, \tag{3.8}$$

where $\mathrm{d}y/\mathrm{d}x$ is the slope of the characteristic. In the following subsections we will learn in more detail how to find the characteristics.

Physically, characteristics bound the region of influence and/or the region of dependence of the solution in a certain point (or region). For example, suppose that a PDE has two characteristics $y_{1,2}$ of the form (3.7) with constant (but different) slopes $\mathrm{d}y_1/\mathrm{d}x > 1$ and $\mathrm{d}y_2/\mathrm{d}x < 1$. Then, Eq. (3.7) describes sets of parallel lines, as shown in Figure 2(a). The solution at some arbitrary point *P*, marked in Figure 2(b), depends only on the solution in the green region. Also, the point *P* influences only the solution in the red region. Note that both the red and the green regions are bounded by the characteristic curves passing through the point *P*. Thus, characteristics describe the directions in which information is distributed, and they also determine the shape and orientation of wave fronts in the solution.

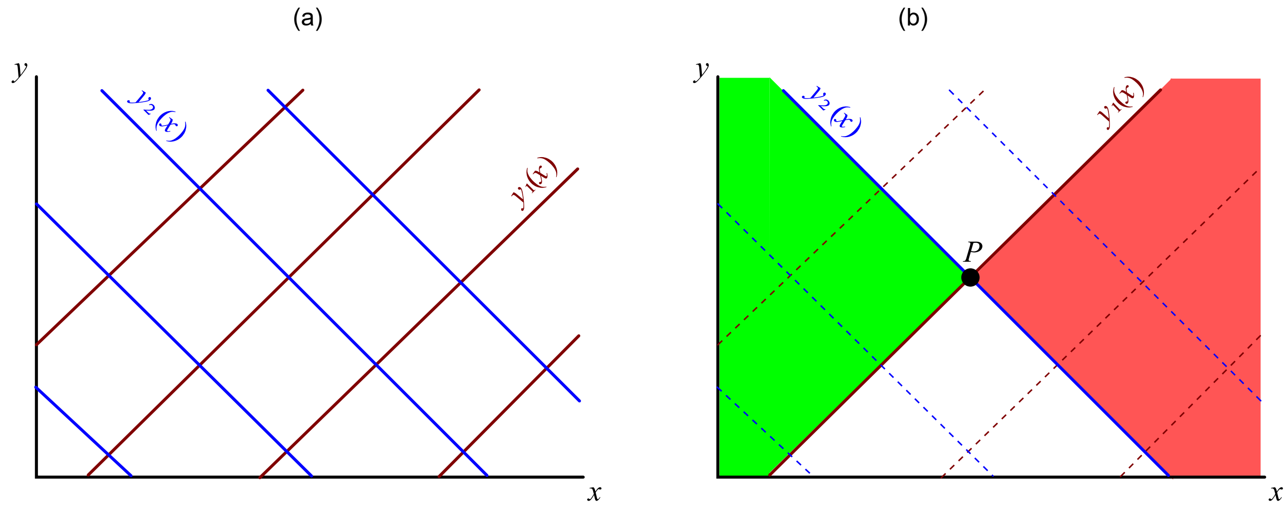


Figure 2: (a) Sketch of two different characteristics with constant slopes. (b) Region of dependence (green) and region of influence (red) of a point *P* for an equation with characteristics y_1 and y_2.

The number of characteristics depends on the type of the (system of) PDE(s). If the type is elliptic, no real characteristics exist. If the type is hyperbolic, the number of real characteristics matches the order of the (system of) PDE(s). If real characteristics exist but their number is lower than the order of the equation or system, then the type is parabolic.

3.6 Finding characteristics of a first-order PDE

Consider a linear first-order PDE with two independent variables

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = f. \tag{3.9}$$

The aim is to replace the partial derivatives in the PDE with a total derivative, e.g., as defined in Eq. (3.8). Thus, we divide Eq. (3.9) by the first coefficient *a* to obtain

$$\frac{\partial u}{\partial x} + \frac{b}{a}\frac{\partial u}{\partial y} = \frac{f}{a}. \tag{3.10}$$

It is evident that the left-hand side of Eq. (3.10) can be replaced by the total derivative defined in Eq. (3.8) only if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a}. \tag{3.11}$$

Integrating Eq. (3.11) we obtain the characteristic

$$y(x) = y_0 + \frac{b}{a}x. \tag{3.12}$$

Replacing the left-hand side of Eq. (3.10) with the total derivative, we obtain the ODE

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{f}{a} \tag{3.13}$$

which describes the variation of the solution *u* along the lines described by Eq. (3.12). Integrating Eq. (3.13) we obtain the analytical solution

$$u\big(x, y(x)\big) = u_0 + \frac{f}{a}x, \tag{3.14}$$

where the integration constant u_0 is determined by initial conditions.

3.7 Finding characteristics of a second-order PDE

Let us now consider a linear second-order PDE with two independent variables. As introduced already in section 3.3, the equation can be written in general form as

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G. \tag{3.15}$$

First, we introduce auxiliary variables

$$P=\frac{\partial u}{\partial x}, \qquad Q=\frac{\partial u}{\partial y}. \tag{3.16}$$

The total differentials of P and Q are defined as

$$\mathrm{d}P=\frac{\partial^2 u}{\partial x^2}\mathrm{d}x+\frac{\partial^2 u}{\partial x\partial y}\mathrm{d}y, \qquad \mathrm{d}Q=\frac{\partial^2 u}{\partial x\partial y}\mathrm{d}x+\frac{\partial^2 u}{\partial y^2}\mathrm{d}y.$$

The task is to replace all partial derivatives in Eq. (3.15) with *P*, *Q*, and a linear combination of total derivatives $\mathrm{d}P/\mathrm{d}x$ and $\mathrm{d}Q/\mathrm{d}y$. The only possibility to reproduce the first and the third term in Eq. (3.15) is to combine the total derivatives as follows:

$$A\frac{\mathrm{d}P}{\mathrm{d}x}+C\frac{\mathrm{d}Q}{\mathrm{d}y}=A\frac{\partial^2 u}{\partial x^2}+\left(A\frac{\mathrm{d}y}{\mathrm{d}x}+C\frac{\mathrm{d}x}{\mathrm{d}y}\right)\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}. \tag{3.17}$$

The first three terms in Eq. (3.15) can be replaced with the right-hand side of Eq. (3.17) only if

$$A\frac{\mathrm{d}y}{\mathrm{d}x}+C\frac{\mathrm{d}x}{\mathrm{d}y}=B. \tag{3.18}$$

Thus, Eq. (3.18) is the defining equation for the slopes of the characteristics, $\mathrm{d}y/\mathrm{d}x$. As in the previous subsection, the characteristics themselves are obtained by integration of $\mathrm{d}y/\mathrm{d}x$. Provided that none of the solutions for $\mathrm{d}y/\mathrm{d}x$ is neither zero nor infinity, we can re-arrange (3.18) to a quadratic equation

$$A\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-B\frac{\mathrm{d}y}{\mathrm{d}x}+C=0. \tag{3.19}$$

Thus, we obtain the criterion introduced already in section 3.3:

- If $B^2-4AC<0$, then no real solutions for $\mathrm{d}y/\mathrm{d}x$ exist. Thus, there are no real characteristics and the PDE is elliptic.
- If $B^2-4AC>0$, the second-order PDE has two characteristics, and thus, it is hyperbolic.
- If $B^2-4AC=0$, the second-order PDE has one characteristic, and thus its type is parabolic.

3.8 Characteristics of a system of linear first-order PDEs

This subsection illustrates how to find characteristics of a system of linear first-order PDEs with two independent variables. The general form of such system was introduced already in subsection 3.4, Eq. (3.3). If characteristics exist, then there exists a linear combination of the PDEs

$$L_i\left(A_{i,j}\frac{\partial u_j}{\partial x}+B_{i,j}\frac{\partial u_j}{\partial y}+C_{i,j}u_j\right)=L_i f_i \tag{3.20}$$

that can be replaced with a linear combination of total differentials

$$m_j\,\mathrm{d}u_j=m_j\left(\frac{\partial u_j}{\partial x}\mathrm{d}x+\frac{\partial u_j}{\partial y}\mathrm{d}y\right). \tag{3.21}$$

Matching the coefficients of partial derivatives on the left-hand side of Eq. (3.20) with those on the right-hand side of Eq. (3.21), we obtain the defining equations for the characteristics as

$$L_i\, A_{i,j} = m_j\, \mathrm{d}x,$$
$$L_i\, B_{i,j} = m_j\, \mathrm{d}y. \tag{3.22}$$

We can eliminate the right-hand sides by subtracting the second set of equations in (3.22) multiplied by d*x* from the first set of equations multiplied by d*y* to obtain the system

$$(\boldsymbol{A}^{\mathrm{T}}\mathrm{d}y - \boldsymbol{B}^{\mathrm{T}}\mathrm{d}x)\,\boldsymbol{L} = \boldsymbol{0}\,. \tag{3.23}$$

The solvability condition leads to Eq. (3.4):

$$\det(\boldsymbol{A}^{\mathrm{T}}\mathrm{d}y - \boldsymbol{B}^{\mathrm{T}}\mathrm{d}x) = 0\,,$$

that can be solved for either $\mathrm{d}y/\mathrm{d}x$ or $\mathrm{d}x/\mathrm{d}y$. The characteristics are then obtained by integration of the slopes as $y(x)$ or $x(y)$, respectively.

In the case of three independent variables, characteristics are sets of surfaces. For a system of first-order linear PDEs with three independent variables, the characteristics are defined by Eq. (3.6), where $\lambda_1, \lambda_2, \lambda_3$, etc. are the components of the normal vector to the characteristic surface.

Note that every linear higher-order PDE can be transformed into a system of first-order PDEs, see, e.g., pp. 2-3 of Schneider (1978) or pp. 27-28 of Chattot (2002). The type and characteristics of the first-order system can then be determined using Eqs. (3.4) or (3.6). However, the first-order system is not always equivalent to the original higher-order PDE, since the transformation can introduce *spurious solutions*, see pp. 57-60 of Wesseling (2001). In other words, all solutions of the original equation also solve the transformed first-order system but not vice versa. The spurious solutions can then alter the type of the system.

3.9 Exercises

a) Consider the equation from Section 3.2:

$$(1 - M_\infty^2)\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0\,, \tag{3.24}$$

that describes the perturbation of a horizontal compressible flow by the presence of a thin profile at zero lift. Show that the equation is elliptic for $M_\infty < 1$ and hyperbolic for $M_\infty > 1$. In the latter case, find the characteristics. What is the physical meaning of the characteristics?

b) The velocity potential ϕ_1, governed by Eq. (3.24), is defined according to Eq. (3.1) such that $u_1 = \partial\phi_1/\partial x$ and $v_1 = \partial\phi_1/\partial y$. Transform the Eq. (3.24) into an equivalent system of two first-order PDEs for u_1 and v_1 as unknowns. Test whether the type of the equivalent first-order system matches the type of the original PDE. *Note:* One of the equations of the equivalent first-order system is a fundamental equation of the potential flow theory.

c) Determine the type of the time-dependent diffusion equation

$$\frac{\partial T}{\partial t} - \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = 0\,. \tag{3.25}$$

Next, transform the equation into an equivalent system of first-order PDEs by introducing auxiliary unknowns $p = \partial T/\partial x$ and $q = \partial T/\partial y$. Find the characteristics if they exist.

d) Convert the Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{3.26}$$

into an equivalent system of first-order equations. Determine the type of the system and find the characteristics if they exist.

e) The one-dimensional propagation of plane acoustic waves in a fluid with density ρ_∞ and speed of sound c_∞ flowing with a velocity u_∞ (relative to the source of the acoustic waves) can be modelled with the following linearized system of equations

Conservation of mass: $$\frac{\partial \rho_1}{\partial t} + u_\infty \frac{\partial \rho_1}{\partial x} + \rho_\infty \frac{\partial u_1}{\partial x} = 0, \tag{3.27a}$$

Conservation of momentum: $$\frac{\partial u_1}{\partial t} + u_\infty \frac{\partial u_1}{\partial x} + \frac{c_\infty^2}{\rho_\infty} \frac{\partial \rho_1}{\partial x} = 0. \tag{3.27b}$$

for the density perturbation ρ_1 and velocity perturbation u_1. Determine the type of the system and find its characteristics if they exist. Interpret your results.

3.10 Literature

More details about the characteristics, the properties of the different types of PDEs, and further examples can be found in:

- Chapter 4 of Chattot (2002),
- Fletcher (1998),
- Chapter 1.2 of the lecture notes for the LVA 302.017 (Kuhlmann, 2021),
- Otto (2011), available as “classification of PDEs.pdf” in Bigarella et al. (2015), part 1 “Computational Fluid Dynamics – Elementary”, section (e) “Partial differential equations – Classification”;
- Part A of Schneider (1978),
- Chapter 2 of Wesseling (2001).

References

Bigarella, E. D. V., Chilvers, J., Demirel, E., & Romanó, F. (2015). *CFD Education Center*. https://cfdblogvienna.blog-spot.com/

Chattot, J.-J. (2002). Partial Differential Equations. In *Computational Aerodynamics and Fluid Dynamics*. Springer-Verlag Berlin.

Fletcher, C. (1998). Partial Differential Equations. In *Computational Techniques for Fluid Dynamics 1: Fundamental and General Techniques*. Springer.

Hosseini, M., Vinuesa, R., Hanifi, A., Henningson, D., & Schlatter, P. (2015, November 22). Video: Turbulent flow around a wing profile, a direct numerical simulation. *68th Annual Meeting of the APS Division of Fluid Dynamics - Gallery of Fluid Motion*. https://doi.org/10.1103/APS.DFD.2015.GFM.V0078

Hosseini, S. M., Vinuesa, R., Schlatter, P., Hanifi, A., & Henningson, D. S. (2016). Direct numerical simulation of the flow around a wing section at moderate Reynolds number. *International Journal of Heat and Fluid Flow*, *61*, 117–128. https://doi.org/10.1016/j.ijheatfluidflow.2016.02.001

Kuhlmann, H. C. (2021). Klassifizierung partieller Differentialgleichungen. In *Numerische Methoden der Strömungs- und Wärmetechnik*. TU Wien [lecture notes for the course No. 302.017]. https://www.fluid.tuwien.ac.at/HendrikKuhl-mann?action=AttachFile&do=get&target=Numerik_I.web.pdf

Moran, J. (1984). *An Introduction to Theoretical and Computational Aerodynamics*. Wiley.

Otto, A. (2011). Classification of PDEs and Related Properties. In *Methods of Numerical Simulation in Fluids and Plas-mas*. University of Alaska [lecture notes]. http://www.mediafire.com/view/qzqfu2cznq556y0/Classifica-tion_of_PDEs.pdf

Schneider, W. (1978). *Mathematische Methoden der Strömungsmechanik*. Vieweg. https://doi.org/10.1007/978-3-322-83943-5

Steinrück, H. (2020). *Asymptotische Methoden der Strömungsmechanik*. TU Wien [lecture notes for the course No. 322.074].

Van Dyke, M. (1982). *An Album of Fluid Motion*. The Parabolic Press.

Wesseling, P. (2001). Classification of partial differential equations. In *Principles of Computational Fluid Dynamics* (1st ed.). Springer. https://doi.org/10.1007/978-3-642-05146-3