6 Subsonic flow over a thin airfoil

6.1 Prandtl-Glauert mapping

The solution for the subsonic flow over a thin airfoil, described by Eqs. (4.22), (4.24), (4.26), (4.30) and (4.32) for $M_\infty < 1$, can again be found with another change of variables, called *Prandtl-Glauert mapping*. The purpose of the mapping is to transform the Eq. (4.22) to a Laplace equation. Thus, the following variables are introduced:

$$\beta = \sqrt{1 - M_\infty^2}\,, \qquad \bar{Y} = \beta Y, \qquad \bar{\phi} = \beta^2 \phi_1\,. \tag{6.1}$$

Substituting the definitions (6.1) into Eq. (4.22) we obtain a Laplace equation in terms of the new variables

$$\frac{\partial^2 \bar{\phi}}{\partial X^2} + \frac{\partial^2 \bar{\phi}}{\partial \bar{Y}^2} = 0\,. \tag{6.2}$$

The flow-tangency conditions (4.30) and (4.32) transform to

$$\begin{aligned} \frac{\partial \bar{\phi}}{\partial \bar{Y}} &= \beta H' && \text{at } \bar{Y} = 0^+ \text{ for } -1/2 < X < 1/2\,, \\ \frac{\partial \bar{\phi}}{\partial \bar{Y}} &= -\beta H' && \text{at } \bar{Y} = 0^- \text{ for } -1/2 < X < 1/2\,. \end{aligned} \tag{6.3}$$

The remaining boundary conditions for $\bar{\phi}$ are the same as Eqs. (4.24) and (4.26) for ϕ_1.

Note that in the limit of an incompressible flow, $M_\infty^2 \to 0$, Eq. (4.22) also reduces to a Laplace equation in the original variables. Thus, Eqs. (6.2), (6.3), (4.24) and (4.26) describe the perturbation of an auxiliary incompressible flow over a profile of modified thickness $\beta\epsilon$. The auxiliary solution $\bar{\phi}$ can therefore be obtained with standard methods for incompressible flows (e.g., with the singularity method), that will be described in the following chapters.

Once $\bar{\phi}(X, \bar{Y})$ is obtained, the solution $\phi_1(X, Y)$ for the perturbation of the compressible flow can be obtained with a backward transformation according to Eq. (6.1). The velocity perturbation components are obtained as follows:

$$\frac{\partial \phi_1}{\partial X} = \frac{1}{\beta^2}\frac{\partial \bar{\phi}}{\partial X}\,, \qquad \frac{\partial \phi_1}{\partial Y} = \frac{1}{\beta}\frac{\partial \bar{\phi}}{\partial \bar{Y}}\,. \tag{6.4}$$

The velocity field of the compressible flow is obtained according to Eqs. (4.33) and (6.4) as

$$U = 1 + \frac{\epsilon}{\beta^2}\frac{\partial \bar{\phi}}{\partial X}\,, \qquad V = \frac{\epsilon}{\beta}\frac{\partial \bar{\phi}}{\partial \bar{Y}}\,. \tag{6.5}$$

The local pressure coefficient is then obtained from Eq. (4.40).

6.2 Prandtl-Glauert rule

It is also common to compute a subsonic flow as incompressible, and afterward make a correction for the compressibility effects. The pressure coefficient of a subsonic flow, C_p, is related to the pressure coefficient C_p^* of the incompressible flow <u>over the same profile</u> as follows:

$$C_p(X, Y) = \frac{C_p^*(X, \beta Y)}{\beta}\,. \tag{6.6}$$

Thus, the pressure perturbation in the compressible flow is larger than in the incompressible flow over the same profile. Also, the pressure perturbation field in the compressible flow is stretched in the *Y*-direction.

6.3 Example

See the solution of the homework on the computation of streamlines, “Subsonic thin profile – streamlines”, provided in TUWEL.

6.4 Literature

- Prandtl et al. (1993), pp. 127-129
- ‘Prandtl-Glauert Mapping’, chapter Bop in Brennen (2004)