322.079 Computational Aerodynamics 5.4.2024

Lecture 4 on 9.4.2024: Subsonic or supersonic compressible flow around a thin airfoil at zero lift

A) Governing equations for compressible potential flow:

- Momentum + Continuity:

$$(c^2-\phi_x^2)\phi_{xx} + (c^2-\phi_y^2)\phi_{yy} - 2\phi_x\phi_y\phi_{xy} = 0 \qquad (1)$$

- Total energy + thermodynamic relationships:

$$c^2 = c_\infty^2 + \frac{\varkappa-1}{2}\left(U^2-\phi_x^2-\phi_y^2\right) \qquad (2)$$

where $\nabla\phi = u$, $\varkappa = \frac{c_p}{c_v}$

Non-linear eq. → can be solved numerically (FEM / FD)

B) Dimensionless form: - for simplification & generality

→ define dimensionless variables:

$$\boxed{\tilde{x} = \frac{x}{L}, \quad \tilde{u} = \frac{u}{U}, \quad \tilde{c} = \frac{c}{c_\infty}}\ ; \quad \tilde{\phi} = \frac{\phi}{UL} \quad \text{s.t.} \quad \tilde{u} = \tilde{\nabla}\tilde{\phi} \qquad (3)$$

→ dimensionless eq.:

$$\left[M_\infty^{-2} + \frac{\varkappa-1}{2}\left(1-\tilde{\phi}_{\tilde{y}}^2\right) - \frac{\varkappa+1}{2}\tilde{\phi}_{\tilde{x}}^2\right]\tilde{\phi}_{\tilde{x}\tilde{x}} + \left[M_\infty^{-2} + \frac{\varkappa-1}{2}\left(1-\tilde{\phi}_{\tilde{x}}^2\right) - \frac{\varkappa+1}{2}\tilde{\phi}_{\tilde{y}}^2\right]\tilde{\phi}_{\tilde{y}\tilde{y}}$$

$$-2\tilde{\phi}_{\tilde{x}}\tilde{\phi}_{\tilde{y}}\tilde{\phi}_{\tilde{x}\tilde{y}} = 0 \qquad (4)$$

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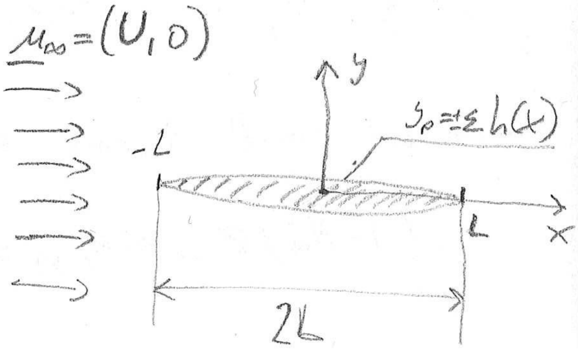
C) Recall the problem formulation:



- Boundary conditions:

— $\tilde{u} \to (1,0)$ as $\tilde{x}^2 + \tilde{y}^2 \to \infty$

$\Longrightarrow \tilde{\phi} = \tilde{x}$ as $\tilde{x}^2 + \tilde{y}^2 \to \infty$ (5)

"far-field"

— symmetry: $\tilde{\phi}_{\tilde{y}} = 0$ at $\tilde{y} = 0$ for $\tilde{x} \le -1$ or $\tilde{x} \ge 1$ (6)

- Kinematic condition:

$$\boxed{\frac{\tilde{v}}{\tilde{u}} = \frac{\tilde{\phi}_{\tilde{y}}}{\tilde{\phi}_{\tilde{x}}} = \pm\varepsilon \frac{\partial \tilde{h}}{\partial \tilde{x}}} \quad \text{at } \tilde{y} = \pm\varepsilon\tilde{h} \text{ for } -1 < \tilde{x} < 1 \qquad (7)$$

D) Asymptotic expansion: (linearization)

- Assumption: the thin profile causes a small perturbation of the incoming flow, proportional to $\varepsilon \ll 1$

$\tilde{\phi} = \phi_0 + \varepsilon\phi_1$ ($\leftarrow$ drop the ~ for brevity) (8)

$\phi_0 = \tilde{x}$ $\leftarrow$ undisturbed flow $u_0 = (1,0)$

$\tilde{\phi} = \tilde{x} + \varepsilon\phi_1$

$\tilde{\phi}_{\tilde{x}} = 1 + \varepsilon\phi_{1,\tilde{x}}$

$(\tilde{\phi}_{\tilde{x}})^2 = 1 + 2\varepsilon\phi_{1,\tilde{x}} + \cancel{\varepsilon^2\phi_{1,\tilde{x}}^2}$

$\tilde{\phi}_{\tilde{x}\tilde{x}} = \varepsilon\phi_{1,\tilde{x}\tilde{x}}$

$\tilde{\phi}_{\tilde{y}} = \varepsilon\phi_{1,\tilde{y}}$

$(\tilde{\phi}_{\tilde{y}})^2 = \cancel{\varepsilon^2\phi_{1,\tilde{y}}^2} \approx 0$

$\tilde{\phi}_{\tilde{y}\tilde{y}} = \varepsilon\phi_{1,\tilde{y}\tilde{y}}$

$\tilde{\phi}_{\tilde{x}\tilde{y}} = \varepsilon\phi_{1,\tilde{x}\tilde{y}}$

$\Rightarrow$ substitute into governing eq. & BCs (drop all ~ for brevity)

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D1) Governing equations:

$$\left[M_\infty^{-2}+\frac{\kappa-1}{2}-\frac{\kappa+1}{2}\left(1+\cancel{2\varepsilon\varnothing_{1,x}}\right)\right]\varepsilon\varnothing_{1,xx}+\left[M_\infty^{-2}+\cancel{\frac{\kappa-1}{2}\cdot\left(-2\varepsilon\varnothing_{1,x}\right)}\right]\varepsilon\varnothing_{1,yy}-2\left(1+\varepsilon\varnothing_{1,x}\right)\varepsilon^2\varnothing_{1,y}\varnothing_{1,xy}=0$$

- cancel small terms $\varepsilon^2 \ll 1$, $\varepsilon^3 \ll 1$ and divide by ε:

$$\boxed{(1-M_\infty^2)\varnothing_{1,xx}+\varnothing_{1,yy}=0}$$ ← linearized eq. for small disturbances $\varepsilon \ll 1$ (9)

⇒ can be solved analytically → $M_\infty > 1$ ⇒ wave equation

→ $M_\infty < 1$ ⇒ Prandtl-Glauert transformation → Laplace eq. ⇒ singularity method

D2) Boundary conditions:

- far-field: $\tilde{\varnothing}=\tilde{x}$ as $\tilde{x}^2+\tilde{y}^2\to\infty$

$$\cancel{\tilde{x}}+\varepsilon\varnothing_1=\cancel{\tilde{x}}$$

$$\varnothing_1=0 \qquad (10)$$

- symmetry: $\tilde{\varnothing}_{\tilde{y}}=0$ at $\tilde{y}=0$ for $\tilde{x}\le -1 \vee \tilde{x}\ge 1$

$$\cancel{\varepsilon}\varnothing_{1,y}=0 \qquad (11)$$

D2, continuation

$\boxed{5.4.2024}$

D2.1 – asymptotic expansion of kinematic BC:

$$\boxed{\phi_y(x, \pm\varepsilon h) = \pm\varepsilon h' \phi_x(x, \pm\varepsilon h)} \qquad (7)$$

- Taylor expansion: $\phi_y(x, \pm\varepsilon h) = \phi_y(x, 0^\pm) \pm \varepsilon h\, \phi_{yy}(x, 0^\pm) + \mathcal{O}(\varepsilon^2)$

$$\phi_x(x, \pm\varepsilon h) = \phi_x(x, 0^\pm) \pm \varepsilon h\, \phi_{xy}(x, 0^\pm) + \mathcal{O}(\varepsilon^2) \qquad (12)$$

- Substitute asymptotic expansion:

$$\phi_y(x, \pm\varepsilon h) \approx \varepsilon \phi_{1,y}(x, 0^\pm) \underbrace{\cancel{\pm \varepsilon h \varepsilon \phi_{1,yy}(x, 0^\pm)}}_{\varepsilon^2 \ll 1}$$

$$\phi_x(x, \pm\varepsilon h) \approx 1 + \varepsilon \phi_{1x}(x, 0^\pm) \cancel{\pm \varepsilon h \varepsilon \phi_{1,xy}(x, 0^\pm)} \qquad (13)$$

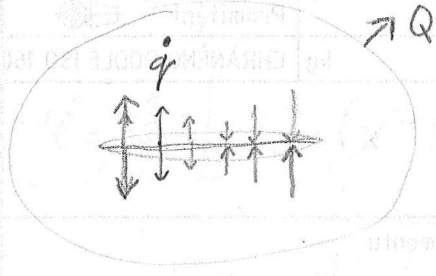
- Substitute (13) into the BC (7):

$$\cancel{\varepsilon} \phi_{1,y}(x, 0^\pm) = \pm \cancel{\varepsilon} h' \left[1 + \cancel{\varepsilon \phi_{1x}(x, 0^\pm)}\right] \quad / : \varepsilon$$

$$\phi_{1,y}(x, 0^\pm) = \pm h' \qquad (14)$$

discontinuous $\frac{\partial \phi_1}{\partial y} = v$ at the airfoil

$\Rightarrow$ For the outer potential flow (excludes thin boundary layer) the airfoil is equivalent to a singular (infinitely thin) source/sink distribution at the airfoil axis.



$\nearrow \dot{Q} = 0$ zero net source $\Rightarrow$ mass is conserved

$\boxed{\text{Flow desribed by } (9, 10, 11, 14)}$

E) Analytical solution for $M_\infty < 1$: Prandtl-Glauert transform

$$\beta = \sqrt{1-M_\infty^2}, \quad \bar{y} = \beta y, \quad \bar{\phi} = \beta^2 \phi_1 \qquad (15)$$

- Governing eq: $\beta^2 \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0 \longrightarrow \frac{\partial^2 \bar{\phi}}{\partial x^2} + \frac{\partial^2 \bar{\phi}}{\partial \bar{y}^2} = 0$

$$\boxed{\bar{\Delta} \bar{\phi} = 0} \qquad (16)$$

- kinematic BC: $\frac{\partial \phi_1}{\partial y} = h' \rightarrow \boxed{\frac{\partial \bar{\phi}}{\partial \bar{y}} = \beta h'}$ $\qquad (17)$

at airfoil

$\Rightarrow$ equivalent incompressible flow ($M_\infty \to 0$) around $\bar{y}_p = \pm \beta \varepsilon h(x)$

F) Solution via source distribution: singularity method

- For an incompressible potential flow due to a line source distribution, an analytical solution exists:

$$\bar{\phi} = \frac{1}{2\pi} \int_{-1}^{1} \bar{q}(\xi) \ln \sqrt{(x-\xi)^2 + \bar{y}^2}\, d\xi \qquad (18) \quad *$$

- The source distribution is given by: $\bar{q}(x) = \bar{\phi}_{\bar{y}}(x, 0^+) - \bar{\phi}_{\bar{y}}(x, 0^-)$

$$= 2 \bar{\phi}_{\bar{y}}(x, 0^+) \qquad (21)$$

Example: consider a parabolic profile: $h = 1 - x^2$

$$h' = -2x \qquad (22)$$

$$\Rightarrow \bar{q}(x) = 2\beta h' = -4\beta x \qquad (23)$$

$$\bar{\phi} = \frac{-2B}{\pi} \int_{-1}^{1} \xi \ln \sqrt{(1-\xi)^2 + \bar{y}^2}\, d\xi \qquad (24)$$

$\rightarrow$ ask Wolfram Mathematica for analytical sol.!

$$\bar{\phi} = \frac{B}{\pi}\left[2x - 2x\bar{y}\left(\arctan\frac{1-x}{\bar{y}} + \arctan\frac{1+x}{\bar{y}}\right) + \left(\bar{y}^2 + 1 - x^2\right)\operatorname{arctanh}\frac{2x}{1+x^2+\bar{y}^2}\right] \qquad (25)$$

$$\bar{u} = \frac{\partial\bar{\phi}}{\partial x} = \frac{1}{2\pi}\int_{-1}^{1} \bar{q}(\xi)\cdot\frac{x-\xi}{(x-\xi)^2 + \bar{y}^2}\, d\xi \qquad (19)$$

$$\bar{u} = \frac{2B}{\pi}\left[2 - \bar{y}\left(\arctan\frac{1-x}{\bar{y}} + \arctan\frac{1+x}{\bar{y}}\right) - x\operatorname{arctanh}\frac{2x}{1+x^2+\bar{y}^2}\right] \qquad (26)$$

$$\bar{v} = \frac{\partial\bar{\phi}}{\partial \bar{y}} = \frac{1}{2\pi}\int_{-1}^{1} \bar{q}(\xi)\,\frac{\bar{y}}{(x-\xi)^2 + \bar{y}^2}\, d\xi \qquad (20)$$

Substitution

$$\bar{v} = \frac{-2B}{\pi}\left[x a + \frac{1}{2}\bar{y}\ln\left(1 - \frac{4x}{(1+x)^2 + \bar{y}^2}\right)\right]; \quad \boxed{a = \arctan\frac{1-x}{\bar{y}} + \arctan\frac{1+x}{\bar{y}}} \qquad (27)$$

G) Backward transformation:

$$\phi_1 = \frac{\bar{\phi}}{\beta^2} = \frac{1}{\beta\pi}\left[2x - 2\beta x y a + \left(\beta^2 y^2 + 1 - x^2\right) b\right] \tag{28}$$

$$\mu_1 = \frac{2}{\beta\pi}\left[2 - \beta y a - x b\right] = \frac{1}{\beta^2}\bar{\mu} \tag{29}$$

$$\nu_1 = \frac{-2}{\pi}\left[x a + \frac{1}{2}\beta y \ln\left(1 - \frac{4x}{(1+x)^2 + \beta^2 y^2}\right)\right] = \frac{1}{\beta}\bar{\nu} \tag{30}$$

322.079 | Dimensional variables (5.4.2024)

H) Pressure:

• Energy equation: $h + \frac{|\underline{u}|^2}{2} = const.$ (31)

• Thermodynamic relationships:
(adiabatic compression of ideal gas)

$$\frac{dT}{T} = \frac{\varkappa - 1}{\varkappa} \frac{dp}{p} \quad (32)$$ (p. 85 of Schneider, 2012)

for $\varepsilon \ll 1$: $T - T_\infty \ll T_\infty$

$p - p_\infty \ll p_\infty$ $\Rightarrow$ $\frac{p - p_\infty}{p_\infty} = \frac{\varkappa}{\varkappa - 1} \frac{T - T_\infty}{T_\infty}$ (33)

From energy eq.: $h - h_\infty = \frac{U^2 - |\underline{u}|^2}{2}$

$$c_p (T - T_\infty) = \frac{U^2 - |\underline{u}|^2}{2}$$

$$\frac{T - T_\infty}{T_\infty} = \frac{U^2 - |\underline{u}|^2}{2 h_\infty}$$ (34)

$$\frac{p - p_\infty}{p_\infty} = \frac{\varkappa}{\varkappa - 1} \frac{U^2 - |\underline{u}|^2}{2 h_\infty}$$

from Brennen (2016), (Bon 12 – Bon 14)

$$h_\infty = \frac{\varkappa}{\varkappa - 1} \frac{p_\infty}{\rho_\infty}$$

$$\frac{p - p_\infty}{p_\infty} = \frac{\rho_\infty}{p_\infty} \frac{U^2 - |\underline{u}|^2}{2} \quad \Big/ \cdot \frac{p_\infty}{\rho_\infty U^2}$$

(Bernoulli equation)

$$\underbrace{\frac{p - p_\infty}{\frac{1}{2} \rho_\infty U^2}}_{\tilde{p}} = \frac{U^2 - |\underline{u}|^2}{2 U^2} \xrightarrow[\text{variables}]{\text{dimensionless}} \tilde{p} = \frac{1}{2}\left(1 - |\tilde{\underline{u}}|^2\right)$$

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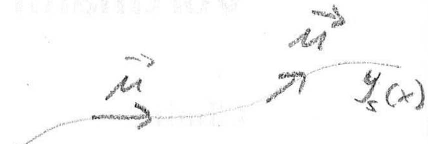
H.1) Bernoulli equation for small disturbances:

$$|\tilde{\underline{u}}|^2 \approx (1+\varepsilon|\underline{u}_1|)^2 = 1 + 2\varepsilon|\underline{u}_1| + \cancel{\varepsilon^2|\underline{u}_1|^2}$$

$$\Rightarrow \boxed{\tilde{p} = -\varepsilon|\underline{u}_1|}$$

I) Streamlines:

- curves tangential to the flow velocity
- trajectories of fluid elements in a steady flow



$$\boxed{\frac{dy_s}{dx} = \frac{v}{u}}$$ ← initial value problem (1st order LODE)

$$\boxed{y_s'(x) = f(x, y_s)}$$ – formal solution:

$$\boxed{y_s(x) = y_0 + \int_{x_0}^{x} v(\bar{x}, y_s(\bar{x}))\, u^{-1}(\bar{x}, y_s(\bar{x}))\, d\bar{x}}$$