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Hess–Smith panel method

1) Distribute singularities along the airfoil surface:

$$\phi = \phi_\infty + \phi_s + \phi_v \qquad (4\text{-}74)$$

unusual sign convention! employed by Moran (1984)

$$= \underbrace{x\cos\alpha + y\sin\alpha}_{\phi_\infty} + \underbrace{\oint \frac{q(s)}{2\pi}\ln r \, ds}_{\phi_s} \underbrace{- \oint \frac{\gamma}{2\pi}\nu \, ds}_{\phi_v}$$

$$u = \underbrace{\cos\alpha}_{u_\infty} + \underbrace{\oint \frac{q(s)}{2\pi}\frac{\partial \ln r}{\partial x} ds}_{u_s} \underbrace{- \oint \frac{\gamma}{2\pi}\frac{\partial \nu}{\partial x} ds}_{u_v}$$

$$= \cos\alpha + \oint \frac{q(s)}{2\pi}\frac{x-x_s}{(x-x_s)^2+(y-y_s)^2} ds + \oint \frac{\gamma}{2\pi}\frac{y-y_s}{(x-x_s)^2+(y-y_s)^2} ds$$

$$v = \underbrace{\sin\alpha}_{v_\infty} + \underbrace{\oint \frac{q(s)}{2\pi}\frac{\partial \ln r}{\partial y} ds}_{v_s} \underbrace{- \oint \frac{\gamma}{2\pi}\frac{\partial \nu}{\partial y} ds}_{v_v}$$

$$= \sin\alpha + \oint \frac{q(s)}{2\pi}\frac{y-y_s}{(x-x_s)^2+(y-y_s)^2} ds - \oint \frac{\gamma}{2\pi}\frac{x-x_s}{(x-x_s)^2+(y-y_s)^2} ds$$

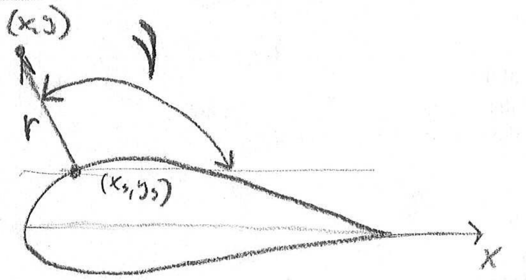


Fig. 4.16, p. 104 in Moran (1984)

$$r = \sqrt{(x-x_s)^2+(y-y_s)^2}$$

$$\nu = \mathrm{arctan2}(y-y_s,\, x-x_s)$$

$$\nu = \begin{cases} \arctan\dfrac{y-y_s}{x-x_s} & \text{if } x \geq 0 \Rightarrow \nu \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle \\ \arctan\left(\dfrac{y-y_s}{x-x_s}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \Rightarrow \nu \in \left(\frac{\pi}{2}; \pi\right\rangle \\ \arctan\left(\dfrac{y-y_s}{x-x_s}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \Rightarrow \nu \in \left(-\pi; -\frac{\pi}{2}\right) \end{cases}$$

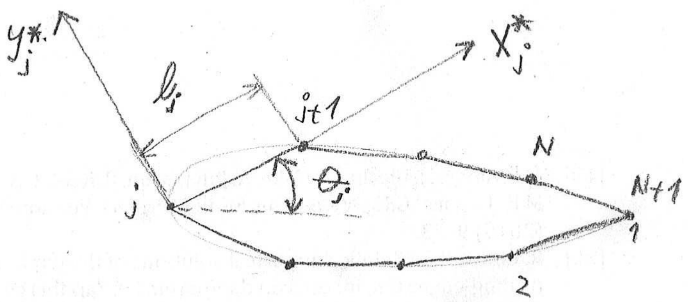
Notation:

x, y – global coordinates of an evaluation point

$x_s(s), y_s(s)$ – coordinates along the surface

s – arclength along the surface

Figs. 4.17 and 4.20:



2) Discretization:

a) N panels, N+1 nodes:

b) Parametrize singularity distributions:

- constant vortex strength, $\boxed{\gamma = \text{const.}}$
- piecewise (panel-wise) source strength, $\boxed{q = q_j, \quad j = 1, \dots, N.}$

c) Replace integrals over the airfoil surface, $\oint ds$, with a sum of line integrals over all panels:

$$\oint ds \rightarrow \sum_{j=1}^{N} \int_0^{l_j} ds_j^*$$

$$\vec{u}(x,y) = \vec{u}_\infty + \sum_{j=1}^{N} q_j \underbrace{\frac{1}{2\pi} \int_0^{l_j} \vec{\nabla}(\ln r)\, ds_j^*}_{\vec{u}_{s_j}(x,y)} + \gamma \sum_{j=1}^{N} \underbrace{-\frac{1}{2\pi} \int_0^{l_j} \vec{\nabla}(\nu)\, ds_j^*}_{\vec{u}_{v_j}(x,y)}$$

where $\vec{u}_\infty = \cos\alpha\, \vec{e}_x + \sin\alpha\, \vec{e}_y$

3) Express the velocity vectors $\vec{u}_{sj}, \vec{u}_{vj}$, the gradients $\vec{\nabla}$, and the integrals $\int_0^{l_j} dx_j^*$ over each panel j in a local coordinate system $x_j^* - y_j^*$ attached to that panel:

- coordinate system for a j-th panel:

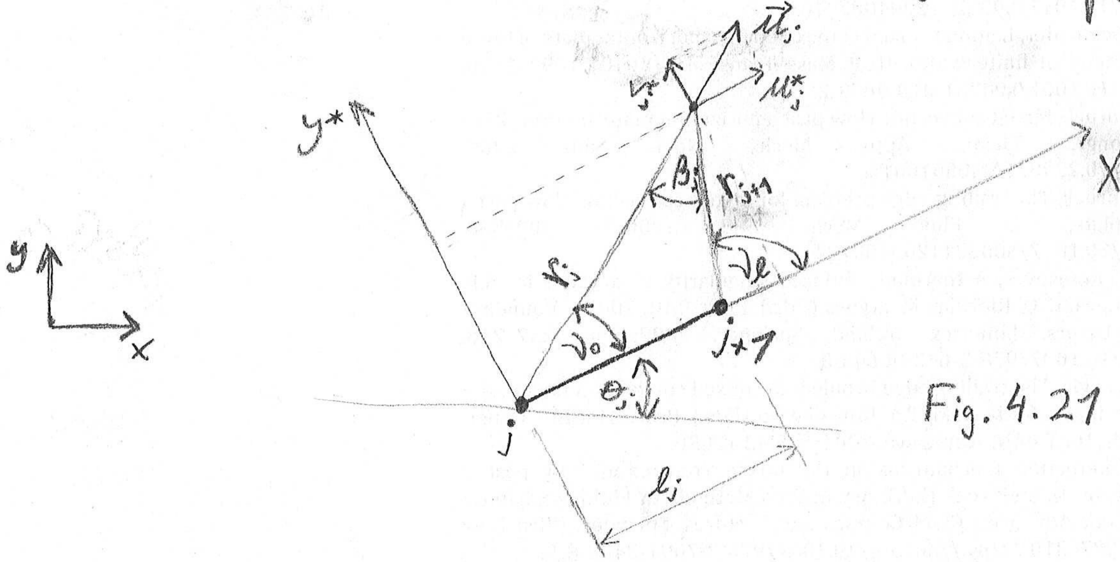


Fig. 4.21

- vectors in $x_j^* - y_j^*$ system:

$$\vec{e}_{x_j^*} = \cos(\theta_j)\,\vec{e}_x + \sin(\theta_j)\,\vec{e}_y$$

$$\vec{e}_{y_j^*} = -\sin(\theta_j)\,\vec{e}_x + \cos(\theta_j)\,\vec{e}_y$$

$$\vec{u}_j = u_j^*\,\vec{e}_{x_j^*} + v_j^*\,\vec{e}_{y_j^*}$$

- gradient in $x_j^* - y_j^*$: $\quad \vec{\nabla} = \vec{e}_{x_j^*}\dfrac{\partial}{\partial x_j^*} + \vec{e}_{y_j^*}\dfrac{\partial}{\partial y_j^*}$

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- Solve the integrals over each panel in their own coordinate systems $x_j^*-y_j^*$:

$$\vec{u}_{sj}(x,y) \equiv u_{sj}^* \vec{e}_{x_j^*} + v_{sj}^* \vec{e}_{y_j^*} = \frac{1}{2\pi}\int_0^{l_j} \vec{\nabla} \ln r \, ds_j^*$$

$$\Rightarrow u_{sj}^* = \frac{1}{2\pi}\int_0^{l_j} \frac{\partial \ln r}{\partial x_j^*} ds_j^* = \frac{-1}{2\pi}\Big[\ln r\Big]_j^{j+1} = \frac{-1}{2\pi} \ln \frac{r_{j+1}}{r_j}$$

$$v_{sj}^* = \frac{1}{2\pi}\int_0^{l_j} \frac{\partial \ln r}{\partial y_j^*} ds_j^* = \frac{1}{2\pi}\Big[\nu\Big]_{s_j^*=0}^{s_j^*=l_j} = \frac{\nu_l - \nu_0}{2\pi} = \frac{\beta_j}{2\pi}$$

Eq. (4-86)

Velocity from vortex distribution:

$$\vec{u}_{vj}(x,y) \equiv u_{vj}^* \vec{e}_{x_j^*} + v_{vj}^* \vec{e}_{y_j^*} = \frac{-1}{2\pi}\int_0^{l_j} \vec{\nabla} \nu \, ds_j^*$$

$$\Rightarrow u_{vj}^* = \frac{-1}{2\pi}\int_0^{l_j} \frac{\partial \nu}{\partial x_j^*} ds_j^* = \frac{1}{2\pi}\Big[\nu\Big]_{s_j^*=0}^{s_j^*=l_j} = \frac{\beta_j}{2\pi} = v_{sj}^*$$

Eq. (4-88)

$$v_{vj}^* = -\frac{1}{2\pi}\int_0^{l_j} \frac{\partial \nu}{\partial y_j^*} ds_j^* = \frac{1}{2\pi}\Big[\ln r\Big]_j^{j+1} = \frac{1}{2\pi} \ln \frac{r_{j+1}}{r_j} = u_{sj}^*$$

 | 4) Transform all velocity vectors to x-y

The net velocity at the evaluation point (x,y) is then obtained as follows:

$$\vec{u} = \vec{u}_\infty + \sum_{j=1}^{N} q_j \left[u^*_{s_j} \vec{e}_{x_j^*} + v^*_{s_j} \vec{e}_{y_j^*} \right] + \gamma \sum_{j=1}^{N} \left[u^*_{v_j} \vec{e}_{x_j^*} + v^*_{v_j} \vec{e}_{y_j^*} \right]$$

$$u \equiv \vec{u} \cdot \vec{e}_x = u_\infty + \sum_{j=1}^{N} q_j \left[u^*_{s_j} \vec{e}_{x_j^*} \cdot \vec{e}_x + v^*_{s_j} \vec{e}_{y_j^*} \cdot \vec{e}_x \right] + \gamma \sum_{j=1}^{N} \left[u^*_{v_j} \vec{e}_{x_j^*} \cdot \vec{e}_x + v^*_{v_j} \vec{e}_{y_j^*} \cdot \vec{e}_x \right]$$

$$\vec{e}_{y_j^*} \cdot \vec{e}_x = -\sin\theta_j \underbrace{\vec{e}_x \cdot \vec{e}_x}_{1} + \cos\theta_j \underbrace{\vec{e}_y \cdot \vec{e}_x}_{0}$$

$$= -\sin\theta_j$$

$$\vec{e}_{x_j^*} \cdot \vec{e}_x = \left(\cos\theta_j \vec{e}_x + \sin\theta_j \vec{e}_y \right) \cdot \vec{e}_x$$

$$= \cos\theta_j \underbrace{\vec{e}_x \cdot \vec{e}_x}_{1} + \sin\theta_j \underbrace{\vec{e}_y \cdot \vec{e}_x}_{0}$$

$$= \cos\theta_j$$

$$\rightarrow u = u_\infty + \sum_{j=1}^{N} q_j \left[u^*_{s_j} \cos\theta_j + v^*_{s_j} (-\sin\theta_j) \right] + \gamma \sum_{j=1}^{N} \left[u^*_{v_j} \cos\theta_j - v^*_{v_j} \sin\theta_j \right]$$

$$= \cos\alpha + \sum_{j=1}^{N} q_j \left[\frac{-1}{2\pi} \ln\frac{r_{j+1}}{r_j} \cos\theta_j - \frac{\beta_j}{2\pi} \sin\theta_j \right] + \gamma \sum_{j=1}^{N} \left[\frac{\beta_j}{2\pi} \cos\theta_j - \frac{1}{2\pi} \ln\frac{r_{j+1}}{r_j} \sin\theta_j \right]$$

See also Eq. (4-89): $u = u^* \underbrace{\cos\theta_j}_{\vec{e}_{x_j^*} \cdot \vec{e}_x} + v^* \underbrace{(-\sin\theta_j)}_{\vec{e}_{y_j^*} \cdot \vec{e}_x}$

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$$V = \vec{u} \cdot \vec{e}_y = \vec{u}_\infty \cdot \vec{e}_y + \sum_{j=1}^{N} q_j \vec{u}_{s_j} \cdot \vec{e}_y + \gamma \sum_{j=1}^{N} \vec{u}_{v_j} \cdot \vec{e}_y$$

$$= V_\infty + \sum_{j=1}^{N} q_j \left[u^*_{s_j} \underbrace{\vec{e}_{x^*_j} \cdot \vec{e}_y}_{\sin\theta_j} + v^*_{s_j} \underbrace{\vec{e}_{y^*_j} \cdot \vec{e}_y}_{\cos\theta_j} \right] + \gamma \sum_{j=1}^{N} \left[u^*_{v_j} \underbrace{\vec{e}_{x^*_j} \cdot \vec{e}_y}_{\sin\theta_j} + v^*_{v_j} \underbrace{\vec{e}_{y^*_j} \cdot \vec{e}_y}_{\cos\theta_j} \right]$$

$$= \sin\alpha + \sum_{j=1}^{N} q_j \left[u^*_{s_j} \sin\theta_j + v^*_{s_j} \cos\theta_j \right] + \gamma \sum_{j=1}^{N} \left[u^*_{v_j} \sin\theta_j + v^*_{v_j} \cos\theta_j \right]$$

$$= \sin\alpha + \sum_{j=1}^{N} q_j \left[\frac{-1}{2\pi} \ln \frac{r_{j+1}}{r_j} \sin\theta_j + \frac{\beta_j}{2\pi} \cos\theta_j \right] + \gamma \sum_{j=1}^{N} \left[\frac{\beta_j}{2\pi} \sin\theta_j + \frac{1}{2\pi} \ln \frac{r_{j+1}}{r_j} \cos\theta_j \right]$$

See also Eq. (4-84)

5) Flow tangency condition

- Prescribed at panel centers as control points:

$$\overline{x}_i = \frac{x_i + x_{i+1}}{2}, \quad \overline{y}_i = \frac{y_i + y_{i+1}}{2}, \quad i = 1, \dots, N \qquad (4\text{-}80)$$

$\Rightarrow$ system of N equations:

$$\boxed{\vec{n}_i \cdot \vec{u}_i \stackrel{!}{=} 0 \qquad \text{for} \qquad i = 1, \dots, N} \qquad (4\text{-}81)$$

where $\vec{n}_i = -\sin\theta_i \, \vec{e}_x + \cos\theta_i \, \vec{e}_y$ and $\vec{u}_i = \vec{u}(\overline{x}_i, \overline{y}_i)$

- Substitute for $\vec{u}_i$ from Eq. (4-83) into Eq. (4-81):

$$\boxed{\vec{n}_i \cdot \vec{u}_i = \vec{n}_i \cdot \vec{u}_\infty + \sum_{j=1}^{N} q_j \, \vec{n}_i \cdot \vec{u}_{sij} + \gamma \sum_{j=1}^{N} \vec{n}_i \cdot \vec{u}_{vij} \stackrel{!}{=} 0} \quad i = 1, \dots, N$$

$\vec{u}_{vij} = \vec{u}_{vj}(\overline{x}_i, \overline{y}_i)$

$\vec{u}_{sij} = \vec{u}_{sj}(\overline{x}_i, \overline{y}_i)$

- Evaluate scalar products:

a) $\vec{n}_i \cdot \vec{u}_\infty = \left[-\sin\theta_i \, \vec{e}_x + \cos\theta_i \, \vec{e}_y\right] \cdot \left[\cos\alpha \, \vec{e}_x + \sin\alpha \, \vec{e}_y\right]$

$$= -\sin\theta_i \cos\alpha + \cos\theta_i \sin\alpha$$

$$= -\sin(\theta_i - \alpha)$$

b) $$\vec{n}_i \cdot \vec{u}_{sij} = \vec{n}_i \cdot \left[u^*_{sij} \vec{e}_{x^*_j} + v^*_{sij} \vec{e}_{y^*_j} \right]$$

$$= u^*_{sij} \, \vec{n}_i \cdot \vec{e}_{x^*_j} + v^*_{sij} \, \vec{e}_{y^*_j} \cdot \vec{n}_i$$

$$= u^*_{sij} \left[-\sin\theta_i \vec{e}_x + \cos\theta_i \vec{e}_y \right] \cdot \left[\cos\theta_j \vec{e}_x + \sin\theta_j \vec{e}_y \right] \dots$$

$$+ v^*_{sij} \left[-\sin\theta_i \vec{e}_x + \cos\theta_i \vec{e}_y \right] \cdot \left[-\sin\theta_j \vec{e}_x + \cos\theta_j \vec{e}_y \right]$$

$$= u^*_{sij} \left(-\sin\theta_i \cos\theta_j + \cos\theta_i \sin\theta_j \right) + v^*_{sij} \left(\sin\theta_i \sin\theta_j + \cos\theta_i \cos\theta_j \right)$$

$$= u^*_{sij} \left[-\sin(\theta_i - \theta_j) \right] + v^*_{sij} \cos(\theta_i - \theta_j)$$

$$= \frac{1}{2\pi} \ln \frac{r_{ij+1}}{r_{ij}} \sin(\theta_i - \theta_j) + \frac{\beta_{ij}}{2\pi} \cos(\theta_i - \theta_j)$$

Alternatively, you can first transform $\vec{u}_{sij}$ to x- and y-directions and afterwards evaluate the projection $\vec{n}_i \cdot \vec{u}_{sij}$:

$$\vec{u}_{sij} = \left[u^*_{sij} \cos\theta_j + v^*_{sij} (-\sin\theta_j) \right] \vec{e}_x + \left[u^*_{sij} \sin\theta_j + v^*_{sij} \cos\theta_j \right] \vec{e}_y$$

$$\vec{n}_i \cdot \vec{u}_{sij} = \left[\left(u^*_{sij} \cos\theta_j - v^*_{sij} \sin\theta_j \right) \vec{e}_x + \left(u^*_{sij} \sin\theta_j + v^*_{sij} \cos\theta_j \right) \vec{e}_y \right] \cdot \left[-\sin\theta_i \vec{e}_x + \cos\theta_i \vec{e}_y \right]$$

$$= \left(u^*_{sij} \cos\theta_j - v^*_{sij} \sin\theta_j \right)(-\sin\theta_i) + \left(u^*_{sij} \sin\theta_j + v^*_{sij} \cos\theta_j \right) \cos\theta_i$$

$$= u^*_{sij} \left(-\cos\theta_j \sin\theta_i + \sin\theta_j \cos\theta_i \right) + v^*_{sij} \left(\sin\theta_j \sin\theta_i + \cos\theta_j \cos\theta_i \right)$$

$$= -u^*_{sij} \sin(\theta_i - \theta_j) + v^*_{sij} \cos(\theta_i - \theta_j)$$

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c) $\vec{n}_i \cdot \vec{u}_{v_{ij}} = u^*_{v_{ij}}\, \vec{n}_i \cdot \vec{e}_{x^*_j} + v^*_{v_{ij}}\, \vec{n}_i \cdot \vec{e}_{y^*_j}$

$$= -u^*_{v_{ij}} \sin(\theta_i - \theta_j) + v^*_{v_{ij}} \cos(\theta_i - \theta_j)$$

$$= -\frac{\beta_{ij}}{2\pi} \sin(\theta_i - \theta_j) + \frac{1}{2\pi} \ln\frac{r_{i,j+1}}{r_{ij}} \cos(\theta_i - \theta_j)$$

• Substitute into the flow tangency condition:

$$\vec{n}_i \cdot \vec{u}_i = \underbrace{-\sin(\theta_i - \alpha)}_{\vec{n}_i \cdot \vec{u}_\infty = -b_i} + \sum_{j=1}^{N} q_j \underbrace{\frac{1}{2\pi}\left[\ln\frac{r_{i,j+1}}{r_{ij}} \sin(\theta_i - \theta_j) + \beta_{ij}\cos(\theta_i - \theta_j)\right]}_{A_{ij} = \vec{n}_i \cdot \vec{u}_{s_{ij}}} \ldots$$

$$+ \gamma \underbrace{\frac{1}{2\pi} \sum_{j=1}^{N}\left[-\beta_{ij} \sin(\theta_i - \theta_j) + \ln\frac{r_{i,j+1}}{r_{ij}} \cos(\theta_i - \theta_j)\right]}_{\sum_{j=1}^{N} \vec{n}_i \cdot \vec{u}_{v_{ij}} = A_{i,N+1}} \stackrel{!}{=} 0$$

for $i = 1, \ldots, N$

• We can write the system of N linear equations as follows:

$$\sum_{j=1}^{N} A_{ij}\, q_j + A_{i,N+1}\, \gamma = b_i \qquad (4\text{-}89)$$

where $A_{ij} = \vec{n}_i \cdot \vec{u}_{s_{ij}} = \frac{1}{2\pi}\left[\ln\frac{r_{i,j+1}}{r_{ij}} \sin(\theta_i - \theta_j) + \beta_{ij}\cos(\theta_i - \theta_j)\right]$ (4-90)

$$A_{i,N+1} = \sum_{j=1}^{N} \vec{n}_i \cdot \vec{u}_{v_{ij}} = \frac{1}{2\pi}\sum_{j=1}^{N}\left[-\beta_{ij}\sin(\theta_i - \theta_j) + \ln\frac{r_{i,j+1}}{r_{ij}}\cos(\theta_i - \theta_j)\right] \qquad (4\text{-}91)$$

$$b_i = -\vec{n}_i \cdot \vec{u}_\infty = \sin(\theta_i - \alpha) \qquad (4\text{-}92)$$

6) Kutta condition

$$\boxed{\vec{t}_1 \cdot \vec{u}_1 + \vec{t}_N \cdot \vec{u}_N \overset{!}{=} 0}$$

(4-82)

where $\vec{t}_i = \cos\theta_i \, \vec{e}_x + \sin\theta_i \, \vec{e}_y$

- Express tangential velocity at control points:

$$\vec{t}_i \cdot \vec{u}_i = \vec{t}_i \cdot \vec{u}_\infty + \sum_{j=1}^{N} q_j \vec{u}_{s_{ij}} \cdot \vec{t}_i + \gamma \sum_{j=1}^{N} \vec{u}_{v_{ij}} \cdot \vec{t}_i \qquad (4\text{-}95)$$

- Express the scalar products:

$$\vec{t}_i \cdot \vec{u}_\infty = \left[\cos\theta_i \, \vec{e}_x + \sin\theta_i \, \vec{e}_y\right] \cdot \left[\cos\alpha \, \vec{e}_x + \sin\alpha \, \vec{e}_y\right]$$

$$= \cos\theta_i \cos\alpha + \sin\theta_i \sin\alpha \qquad (4\text{-}95)$$

$$= \cos(\theta_i - \alpha)$$

$$\vec{t}_i \cdot \vec{u}_{s_{ij}} = \left[\cos\theta_i \, \vec{e}_x + \sin\theta_i \, \vec{e}_y\right] \cdot \left[u^*_{s_{ij}} \, \vec{e}_{x^*_j} + v^*_{s_{ij}} \, \vec{e}_{y^*_j}\right] \qquad (4\text{-}95)$$

$$= u^*_{s_{ij}} \, \vec{t}_i \cdot \vec{e}_{x^*_j} + v^*_{s_{ij}} \, \vec{t}_i \cdot \vec{e}_{y^*_j}$$

$$= u^*_{s_{ij}} \left[\cos\theta_i \, \vec{e}_x + \sin\theta_i \, \vec{e}_y\right] \cdot \left[\cos\theta_j \, \vec{e}_x + \sin\theta_j \, \vec{e}_y\right] + v^*_{s_{ij}} \left[\cos\theta_i \, \vec{e}_x + \sin\theta_i \, \vec{e}_y\right] \cdot \left[-\sin\theta_j \, \vec{e}_x + \cos\theta_j \, \vec{e}_y\right]$$

$$= u^*_{s_{ij}} \left(\cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j\right) + v^*_{s_{ij}} \left(-\cos\theta_i \sin\theta_j + \sin\theta_i \cos\theta_j\right)$$

$$= u^*_{s_{ij}} \cos(\theta_i - \theta_j) + v^*_{s_{ij}} \sin(\theta_i - \theta_j) = -\frac{1}{2\pi} \ln \frac{r_{ij+1}}{r_{ij}} \cos(\theta_i - \theta_j) + \frac{\beta_{ij}}{2\pi} \sin(\theta_i - \theta_j)$$

$$\vec{t}_i \cdot \vec{\mu}_{vij} = \mu^*_{vij}\, \vec{t}_i \cdot \vec{e}_{x^*_j} + v^*_{vij}\, \vec{t}_i \cdot \vec{e}_{y^*_j}$$

$$= \mu^*_{vij} \cos(\theta_i - \theta_j) + v^*_{vij} \sin(\theta_i - \theta_j)$$

$$= \frac{\beta_{ij}}{2\pi} \cos(\theta_i - \theta_j) + \frac{1}{2\pi} \ln\frac{r_{ij+1}}{r_{ij}} \sin(\theta_i - \theta_j) \qquad (4-95)$$

- Substitute into the kutta condition:

$$\vec{t}_1 \cdot \vec{u}_\infty + \vec{t}_N \cdot \vec{u}_\infty + \sum_{j=1}^N q_j \underbrace{\left[\vec{u}_{s1j} \cdot \vec{t}_1 + \vec{u}_{sNj} \cdot \vec{t}_N\right]} + \gamma \sum_{j=1}^N \left[\vec{u}_{v1j} \cdot \vec{t}_1 + \vec{u}_{vNj} \cdot \vec{t}_N\right] \stackrel{!}{=} 0$$

$$\underbrace{\cos(\theta_1 - \alpha) + \cos(\theta_N - \alpha)}_{-b_{N+1}} + \sum_{j=1}^N q_j \underbrace{\frac{1}{2\pi}\left[-\ln\frac{r_{1j+1}}{r_{1j}} \cos(\theta_1 - \theta_j) - \ln\frac{r_{Nj+1}}{r_{Nj}} \cos(\theta_N - \theta_j) + \frac{\beta_{1j}}{2\pi} \sin(\theta_1 - \theta_j) + \frac{\beta_{Nj}}{2\pi} \sin(\theta_N - \theta_j)\right]}_{A_{N+1,j}}$$

$$+ \gamma \underbrace{\frac{1}{2\pi} \sum_{j=1}^N \left[\beta_{1j} \cos(\theta_1 - \theta_j) + \beta_{Nj} \cos(\theta_N - \theta_j) + \ln\frac{r_{1j+1}}{r_{1j}} \sin(\theta_1 - \theta_j) + \ln\frac{r_{Nj+1}}{r_{Nj}} \sin(\theta_N - \theta_j)\right]}_{A_{N+1,N+1}} \stackrel{!}{=} 0$$

- Write the (N+1)th linear equation in the following form:

$$\sum_{j=1}^N A_{N+1,j}\, q_j + A_{N+1,N+1}\, \gamma = b_{N+1} \qquad (4-93)$$

$$A_{N+1,j} = \vec{u}_{s1j} \cdot \vec{t}_1 + \vec{u}_{sNj} \cdot \vec{t}_N = \frac{1}{2\pi}\left[-\ln\frac{r_{1j+1}}{r_{1j}} \cos(\theta_1 - \theta_j) - \ln\frac{r_{Nj+1}}{r_{Nj}} \cos(\theta_N - \theta_j) + \frac{\beta_{1j}}{2\pi} \sin(\theta_1 - \theta_j) + \frac{\beta_{Nj}}{2\pi} \sin(\theta_N - \theta_j)\right]$$

$$A_{N+1,N+1} = \sum_{j=1}^N \left(\vec{u}_{v1j} \cdot \vec{t}_1 + \vec{u}_{vNj} \cdot \vec{t}_N\right) = \frac{1}{2\pi} \sum_{j=1}^N \left[\beta_{1j} \cos(\theta_1 - \theta_j) + \beta_{Nj} \cos(\theta_N - \theta_j) + \ln\frac{r_{1j+1}}{r_{1j}} \sin(\theta_1 - \theta_j) + \ln\frac{r_{Nj+1}}{r_{Nj}} \sin(\theta_N - \theta_j)\right]$$

$$b_{N+1} = -\cos(\theta_1 - \alpha) - \cos(\theta_N - \alpha) \qquad (4-94)$$