TECHNISCHE UNIVERSITÄT WIEN

Compressible potential flow

Computational Aerodynamics

Governing equations of fluid flow

Governing equations of fluid flow

- Conservation of mass:

rate of change of mass in a <u>fixed</u> control volume

net mass outflow through the boundaries of the control volume

$$\frac{\partial}{\partial t}\iiint_V \rho \, \mathrm{d}V + \oiint_S \rho \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} = 0$$

integral form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0$$

differential form

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

change of density of a fluid parcel

change of volume of the fluid parcel

Material derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}$$

Governing equations of fluid flow

Remember the Newton's 2nd law of motion?

$$\boxed{m\boldsymbol{a} = \boldsymbol{F}}$$

- Conservation of mass: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$
- Conservation of momentum: $\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p_A + \nabla \cdot \underline{\underline{\boldsymbol{\tau}}} + \rho \boldsymbol{f}$

rate of change of momentum of a fluid parcel

forces acting on the parcel

absolute pressure

friction

(external) body force - usually gravity $\boldsymbol{g}$

(internal) surface forces

Common decomposition when $\boldsymbol{f} = \boldsymbol{g}$ and ρ = const.

hydrostatic

$$p_A = p_h + p_G$$

relative/gauge

$$\nabla p_h = \rho \boldsymbol{g}$$

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p_G + \nabla \cdot \underline{\underline{\boldsymbol{\tau}}}$$

Material derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla$$

Governing equations of fluid flow

Material derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}$$

- Conservation of mass: $$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0$$
- Conservation of momentum: $$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} p_A + \boldsymbol{\nabla} \cdot \underline{\underline{\boldsymbol{\tau}}} + \rho \boldsymbol{f}$$
- Conservation of energy: $$\rho \frac{\mathrm{D}}{\mathrm{D}t}\underbrace{\left(\underbrace{\frac{|\boldsymbol{u}|^2}{2}}_{\text{kinetic energy}} + \underbrace{e}_{\text{internal energy}}\right)}_{\text{total energy}} = \boldsymbol{\nabla} \cdot \left(\overbrace{\underline{\underline{\boldsymbol{\tau}}}\boldsymbol{u}}^{\text{dissipation of momentum}} - \underbrace{p_A \boldsymbol{u}}_{\text{work of pressure}}\right) + \overbrace{\rho \boldsymbol{f} \cdot \boldsymbol{u}}^{\text{work of body forces}} + \underbrace{\boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} T)}_{\text{heat conduction}} + \overbrace{q_H}^{\text{internal heating}}$$

internal heating:
chemical reactions,
radiation,
electric curent, …

Governing equations of fluid flow

- Conservation of mass: $$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0$$
  (continuity equation)

Schlichting & Gersten (2017), p.52

Material derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}$$

- Conservation of momentum: $$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} p_A + \boldsymbol{\nabla} \cdot \underline{\underline{\boldsymbol{\tau}}} + \rho \boldsymbol{f}$$

S & G, p.68

- Conservation of energy: $$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{|\boldsymbol{u}|^2}{2} + e\right) = \boldsymbol{\nabla} \cdot \left(\underline{\underline{\boldsymbol{\tau}}}\boldsymbol{u} - p_A \boldsymbol{u}\right) + \rho \boldsymbol{f} \cdot \boldsymbol{u} + \boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} T) + q_H$$

S & G, p.71

- Constitutive equations:

Material equations

for Newtonian fluids:

$$\underline{\underline{\tau}} = \mu \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T - \frac{2}{3} \mathbb{I} (\boldsymbol{\nabla} \cdot \boldsymbol{u}) \right)$$

Thermodynamic relationships

For ideal gas:

$$p = \rho R T$$

$$\mathrm{d}e = c_v \, \mathrm{d}T$$

Energy equation

Total energy

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{|\boldsymbol{u}|^2}{2} + e\right) = \boldsymbol{\nabla} \cdot \left(\underline{\underline{\boldsymbol{\tau}}}\boldsymbol{u} - p_A \boldsymbol{u}\right) + \rho \boldsymbol{f} \cdot \boldsymbol{u} + \boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} T)$$

Internal energy

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = \underline{\underline{\boldsymbol{\tau}}} : \nabla \mathbf{u} - p_A \boldsymbol{\nabla} \cdot \boldsymbol{u} + \boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} T)$$

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} - \frac{p_A}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = \underline{\underline{\boldsymbol{\tau}}} : \nabla \mathbf{u} + \boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} T)$$

$$\rho T \frac{\mathrm{D}s}{\mathrm{D}t} = \rho \frac{\mathrm{D}h}{\mathrm{D}t} - \frac{\mathrm{D}p_A}{\mathrm{D}t} = \underline{\underline{\boldsymbol{\tau}}} : \nabla \mathbf{u} + \boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} T)$$

Temperature form

$$\rho c_p \frac{\mathrm{D}T}{\mathrm{D}t} - \beta T \frac{\mathrm{D}p_A}{\mathrm{D}t} = \underline{\underline{\boldsymbol{\tau}}} : \nabla \mathbf{u} + \boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} T)$$

Momentum

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} p_A + \boldsymbol{\nabla} \cdot \underline{\underline{\boldsymbol{\tau}}} + \rho \boldsymbol{f} \qquad | \cdot \boldsymbol{u}$$

Mechanical energy

$$\frac{\rho}{2}\frac{\mathrm{D}|\boldsymbol{u}|^2}{\mathrm{D}t} = -\boldsymbol{u} \cdot \boldsymbol{\nabla} p_A + \boldsymbol{u} \cdot \left(\boldsymbol{\nabla} \cdot \underline{\underline{\boldsymbol{\tau}}}\right) + \rho \boldsymbol{f} \cdot \boldsymbol{u}$$

Mass conservation,

$$\nabla \cdot \boldsymbol{u} = -\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t},$$

Entropy

$$\frac{\mathrm{D}h}{\mathrm{D}t} - \frac{1}{\rho}\frac{\mathrm{D}p_A}{\mathrm{D}t} = T\frac{\mathrm{D}s}{\mathrm{D}t}$$

Enthalpy

$$h = e + \frac{p_A}{\rho}, \quad \frac{\mathrm{D}h}{\mathrm{D}t} = \frac{\mathrm{D}e}{\mathrm{D}t} + \frac{1}{\rho}\frac{\mathrm{D}p_A}{\mathrm{D}t} - \frac{p_A}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t}$$

For ideal gas:

$$h = h(p_A, T) \Longrightarrow \frac{\mathrm{D}h}{\mathrm{D}t} = \underbrace{\frac{1 - \beta T}{\rho}}_{\partial h/\partial p}\frac{\mathrm{D}p_A}{\mathrm{D}t} + \underbrace{c_p}_{\partial h/\partial T}\frac{\mathrm{D}T}{\mathrm{D}t}$$

Dimensionless variables

Dimensionless variables

- Refer variables to some representative values for the given problem

$$\tilde{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \quad \tilde{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U}, \quad \tilde{t} = \frac{t}{t^*}, \quad \tilde{\rho} = \frac{\rho}{\rho_0}, \quad \tilde{p} = \frac{p}{p_0}, \quad \tilde{\mu} = \frac{\mu}{\mu_0}, \dots \quad \sim \mathcal{O}(1)$$

- Substitute into the governing equations

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0 \longrightarrow \frac{\rho_0}{t^*}\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\rho_0 U}{L}\tilde{\nabla} \cdot (\tilde{\rho}\tilde{\boldsymbol{u}}) = 0 \quad \Big| \times \frac{L}{\rho_0 U}$$

- Rearrange prefactors into dimensionless groups

$$\underbrace{\frac{L}{U t^*}}_{\text{Str}}\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\tilde{\rho}\tilde{\boldsymbol{u}}) = 0$$

- Compare orders of magnitude of different terms

If t^* of a <u>time-dependent</u> flow is not known *a priori*:

Convective time scale

$$t^* = \frac{L}{U}$$

Viscous time scale

$$t^* = \frac{L^2 \rho_0}{\mu_0}$$

Criterion for steady flows

$$\text{Str} \ll 1 \quad \Longleftrightarrow \quad t^* \gg \frac{L}{U}$$

Dimensionless momentum equation

- Refer variables to some representative values for the given problem

$$\tilde{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \qquad \tilde{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U}, \qquad \tilde{t} = \frac{t}{t^*}, \qquad \tilde{\rho} = \frac{\rho}{\rho_0}, \qquad \tilde{p} = \frac{p}{p_0}, \qquad \tilde{\mu} = \frac{\mu}{\mu_0}, \dots \qquad \sim \mathcal{O}(1)$$

- Recall the momentum equation

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p_A + \nabla \cdot \mu \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T - \frac{2}{3} \mathbb{I} (\nabla \cdot \boldsymbol{u}) \right) + \rho \boldsymbol{g}$$

- The dimensionless form reads

$$\mathrm{Str}\ \tilde{\rho} \frac{\partial \tilde{\boldsymbol{u}}}{\partial \tilde{t}} + \tilde{\rho} \tilde{\boldsymbol{u}} \cdot \tilde{\nabla} \tilde{\boldsymbol{u}} = -\,\mathrm{Eu}\quad \tilde{\nabla} \tilde{p} + \frac{1}{\mathrm{Re}} \quad \tilde{\nabla} \cdot \tilde{\mu} \left(\tilde{\nabla} \tilde{\boldsymbol{u}} + \tilde{\nabla} \tilde{\boldsymbol{u}}^T - \frac{2}{3} \mathbb{I} (\tilde{\nabla} \cdot \tilde{\boldsymbol{u}}) \right) - \frac{1}{\mathrm{Fr}^2}\ \tilde{\rho} \boldsymbol{e}_y$$

Task for you

Find the definitions of the dimensionless numbers

Dimensionless momentum equation

- Refer variables to some representative values for the given problem

$$\tilde{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \qquad \tilde{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U}, \qquad \tilde{t} = \frac{t}{t^*}, \qquad \tilde{\rho} = \frac{\rho}{\rho_0}, \qquad \tilde{p} = \frac{p}{p_0}, \qquad \tilde{\mu} = \frac{\mu}{\mu_0}, \dots \qquad \sim \mathcal{O}(1)$$

- Recall the momentum equation

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p_A + \nabla \cdot \mu \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T - \frac{2}{3} \mathbb{I} (\nabla \cdot \boldsymbol{u}) \right) + \rho \boldsymbol{g}$$

Define

$$\frac{\tilde{\mathrm{D}}}{\mathrm{D}\tilde{t}} = \mathrm{Str} \frac{\partial}{\partial \tilde{t}} + \tilde{\boldsymbol{u}} \cdot \tilde{\nabla}$$

- The dimensionless form reads

$$\frac{L}{U t^*} \tilde{\rho} \frac{\partial \tilde{\boldsymbol{u}}}{\partial \tilde{t}} + \tilde{\rho} \tilde{\boldsymbol{u}} \cdot \tilde{\nabla} \tilde{\boldsymbol{u}} = - \boxed{\quad} \tilde{\nabla} \tilde{p} + \boxed{\quad} \tilde{\nabla} \cdot \tilde{\mu} \left(\tilde{\nabla} \tilde{\boldsymbol{u}} + \tilde{\nabla} \tilde{\boldsymbol{u}}^T - \frac{2}{3} \mathbb{I} (\tilde{\nabla} \cdot \tilde{\boldsymbol{u}}) \right) - \boxed{\quad} \tilde{\rho} \boldsymbol{e}_y$$

$$\mathrm{Str} \; \tilde{\rho} \frac{\partial \tilde{\boldsymbol{u}}}{\partial \tilde{t}} + \tilde{\rho} \tilde{\boldsymbol{u}} \cdot \tilde{\nabla} \tilde{\boldsymbol{u}} = - \mathrm{Eu} \; \tilde{\nabla} \tilde{p} + \frac{1}{\mathrm{Re}} \; \tilde{\nabla} \cdot \tilde{\mu} \left(\tilde{\nabla} \tilde{\boldsymbol{u}} + \tilde{\nabla} \tilde{\boldsymbol{u}}^T - \frac{2}{3} \mathbb{I} (\tilde{\nabla} \cdot \tilde{\boldsymbol{u}}) \right) - \frac{1}{\mathrm{Fr}^2} \; \tilde{\rho} \boldsymbol{e}_y$$

Dimensionless momentum equation

- Refer variables to some representative values for the given problem

$$\tilde{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \qquad \tilde{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U}, \qquad \tilde{t} = \frac{t}{t^*}, \qquad \tilde{\rho} = \frac{\rho}{\rho_0}, \qquad \tilde{p} = \frac{p}{p_0}, \qquad \tilde{\mu} = \frac{\mu}{\mu_0}, \dots \qquad \sim \mathcal{O}(1)$$

- The dimensionless momentum equation reads

$$\frac{L}{Ut^*}\,\tilde{\rho}\,\frac{\partial \tilde{\boldsymbol{u}}}{\partial \tilde{t}} + \tilde{\rho}\tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{\boldsymbol{u}} = -\frac{p_0}{\rho_0 U^2}\tilde{\nabla}\tilde{p} + \frac{\mu_0}{\rho_0 U L}\tilde{\nabla} \cdot \tilde{\mu}\left(\tilde{\nabla}\tilde{\boldsymbol{u}} + \tilde{\nabla}\tilde{\boldsymbol{u}}^T - \frac{2}{3}\mathbb{I}(\tilde{\nabla} \cdot \tilde{\boldsymbol{u}})\right) - \frac{gL}{U^2}\,\tilde{\rho}\boldsymbol{e}_y$$

- For aerodynamics: $p_0 = \rho_0 U^2 =$ 2× stagnation pressure

$$\tilde{\rho}\frac{\tilde{\mathrm{D}}\tilde{\boldsymbol{u}}}{\mathrm{D}\tilde{t}} = \quad -\tilde{\nabla}\tilde{p} \qquad + \frac{1}{\mathrm{Re}}\,\tilde{\nabla} \cdot \tilde{\mu}\left(\tilde{\nabla}\tilde{\boldsymbol{u}} + \tilde{\nabla}\tilde{\boldsymbol{u}}^T - \frac{2}{3}\mathbb{I}(\tilde{\nabla} \cdot \tilde{\boldsymbol{u}})\right) - \frac{1}{\mathrm{Fr}^2}\,\tilde{\rho}\boldsymbol{e}_y$$

- Compare orders of magnitude for a given problem

Consider a small glider: $L \gtrsim 1\mathrm{m}$, $U \gtrsim 10\mathrm{m/s}$

in air: $\mu_0 \sim 10^{-5}\,\frac{\mathrm{kg}}{\mathrm{m\,s}}$, $g \sim 10\,\frac{\mathrm{m}}{\mathrm{s}^2}$, $\rho_0 \sim 1\,\frac{\mathrm{kg}}{\mathrm{m}^3}$

$\Longrightarrow$ $\mathrm{Re} \gtrsim 10^6 \gg 1$, $\mathrm{Fr}^2 \gtrsim 10 \gg 1$

Dimensionless energy equation

$$\rho c_p \frac{\mathrm{D}T}{\mathrm{D}t} - \beta T \frac{\mathrm{D}p_A}{\mathrm{D}t} = \underline{\underline{\boldsymbol{\tau}}} : \nabla \mathbf{u} + \boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} T)$$

- Refer variables to some reference values for the given problem

$$\tilde{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \quad \tilde{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U}, \quad \tilde{T} = \frac{T}{T^*}, \quad \tilde{t} = \frac{t}{t^*}, \quad \tilde{\rho} = \frac{\rho}{\rho_0}, \quad \tilde{p} = \frac{p}{\rho_0 U^2}, \quad \tilde{\mu} = \frac{\mu}{\mu_0}, \quad \tilde{c}_p = \frac{c_p}{c_p^*}, \quad \tilde{k} = \frac{k}{k_0},$$

$$\tilde{\beta} = \frac{\beta}{\beta_0}$$

$$\tilde{\rho}\tilde{c}_p \frac{\tilde{\mathrm{D}}\tilde{T}}{\mathrm{D}\tilde{t}} - \frac{\beta_0 U^2}{c_p^*} \tilde{\beta}\tilde{T} \frac{\tilde{\mathrm{D}}\tilde{p}}{\mathrm{D}\tilde{t}} = \underbrace{\frac{\mu_0 U}{\rho_0 c_p^* L T^*}}_{\mathrm{Br}/\mathrm{Pe}} \underline{\underline{\tilde{\boldsymbol{\tau}}}} : \tilde{\nabla}\tilde{\mathbf{u}} + \underbrace{\frac{k_0}{\rho_0 c_p^* L U}}_{\mathrm{Pe}^{-1}} \tilde{\boldsymbol{\nabla}} \cdot (\tilde{k} \tilde{\boldsymbol{\nabla}} \tilde{T})$$

Stagnation temperature

$$T^* = 2(T_t - T_0) = \frac{U^2}{c_p^*} \gtrsim 10^{-2}$$

- Compare orders of magnitude

$$L \gtrsim 1\mathrm{m}, \quad U \gtrsim 10\,\frac{\mathrm{m}}{\mathrm{s}}, \quad \rho_0 \sim 1\,\frac{\mathrm{kg}}{\mathrm{m}^3},$$

$$\mu_0 \sim 10^{-5}\,\frac{\mathrm{kg}}{\mathrm{m\,s}}, \quad k_0 \sim 10^{-2}\,\frac{\mathrm{W}}{\mathrm{m\,K}},$$

$$c_p^* \sim 10^3\,\frac{\mathrm{J}}{\mathrm{kg\,K}}, \quad \beta_0 \sim 10^{-3}\mathrm{K}^{-1},$$

$$\Longrightarrow$$

$$\mathrm{Pe} \gtrsim 10^6 \gg 1 \qquad \frac{\mathrm{Br}}{\mathrm{Pe}} \lesssim 10^{-4}$$

$$10^{-4} \lesssim \frac{\beta_0 U^2}{c_p^*} \lesssim 10^{-2}$$

Potential flow

Inviscid flow

$$\mathrm{Str}\,\tilde{\rho}\frac{\partial\tilde{\boldsymbol{u}}}{\partial\tilde{t}}+\tilde{\rho}\,\tilde{\boldsymbol{u}}\cdot\tilde{\nabla}\tilde{\boldsymbol{u}}=\tilde{\nabla}\tilde{p}+\frac{1}{\mathrm{Re}}\tilde{\nabla}\cdot\tilde{\mu}\left(\tilde{\nabla}\tilde{\boldsymbol{u}}+\tilde{\nabla}\tilde{\boldsymbol{u}}^{T}-\frac{2}{3}\mathbb{I}(\tilde{\nabla}\cdot\tilde{\boldsymbol{u}})\right)-\frac{1}{\mathrm{Fr}^2}\,\tilde{\rho}\boldsymbol{e}_y$$

- Compare orders of magnitude for a given problem

Consider a small glider: $L \gtrsim 1\mathrm{m}$, $U \gtrsim 10\mathrm{m/s}$

in air: $\mu_0 \sim 10^{-5}\,\frac{\mathrm{kg}}{\mathrm{m\,s}}$, $g \sim 10\,\frac{\mathrm{m}}{\mathrm{s}^2}$, $\rho_0 \sim 1\,\frac{\mathrm{kg}}{\mathrm{m}^3}$

$\Longrightarrow$ $\mathrm{Re} \gtrsim 10^6 \gg 1$, $\mathrm{Fr}^2 \gtrsim 10 \gg 1$

inviscid $\Downarrow$ $\mathrm{Re} \to \infty$

Euler equations

conservation of mass: $\mathrm{Str}\frac{\partial\tilde{\rho}}{\partial\tilde{t}}+\tilde{\nabla}\cdot(\tilde{\rho}\,\tilde{\boldsymbol{u}})=0$

conservation of momentum: $\mathrm{Str}\frac{\partial\tilde{\boldsymbol{u}}}{\partial\tilde{t}}+(\tilde{\boldsymbol{u}}\cdot\tilde{\nabla})\tilde{\boldsymbol{u}}=-\frac{1}{\tilde{\rho}}\tilde{\nabla}\tilde{p}$

! Too many unknowns: ρ, $\boldsymbol{u}$, p

ρ and p related through the speed of sound

$$c=\sqrt{\left(\frac{\partial p}{\partial\rho}\right)_s}$$

Isentropic flow

conservation of mass: $\mathrm{Str}\frac{\partial\tilde{\rho}}{\partial\tilde{t}} + \tilde{\nabla}\cdot(\tilde{\rho}\,\tilde{\boldsymbol{u}}) = 0$

conservation of momentum: $\mathrm{Str}\frac{\partial\tilde{\boldsymbol{u}}}{\partial\tilde{t}} + (\tilde{\boldsymbol{u}}\cdot\tilde{\nabla})\tilde{\boldsymbol{u}} = -\frac{1}{\tilde{\rho}}\tilde{\nabla}\tilde{p}$

ρ and p related through the speed of sound

$$c = \sqrt{\left(\frac{\partial p}{\partial\rho}\right)_s}$$

Pe $\longrightarrow \infty$

$$\rho T\frac{\mathrm{D}s}{\mathrm{D}t} = \cancel{\underline{\boldsymbol{\tau}}:\nabla\mathbf{u}} + \cancel{\boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}T)}$$

$$\frac{\mathrm{Br}}{\mathrm{Pe}} \to 0$$

$\Longrightarrow$

For isentropic flow (s = const.):

$$\left(\frac{\partial p}{\partial\rho}\right)_s = \frac{\mathrm{d}p}{\mathrm{d}\rho} = c^2$$

$$\nabla p = c^2\nabla\rho$$

$$\tilde{\nabla}\tilde{p} = \underbrace{\frac{c_\infty^2}{U^2}}_{M_\infty^{-2}}\tilde{c}^2\tilde{\nabla}\tilde{\rho}$$

Potential flow

Consider a steady state: $\mathrm{Str} \to 0$

conservation of mass: $\rho\nabla\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\rho = 0 \;\; | \div \rho$

conservation of momentum: $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\dfrac{c^2}{\rho}\nabla\rho$

$$\nabla\cdot\boldsymbol{u} - \frac{1}{c^2}\boldsymbol{u}\cdot[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}] = 0$$

In 2D after rearrangement:

$$(c^2 - u^2)\frac{\partial u}{\partial x} + (c^2 - v^2)\frac{\partial v}{\partial y} - uv\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$$\left(c^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right)\frac{\partial^2\phi}{\partial x^2} + \left(c^2 - \left(\frac{\partial\phi}{\partial y}\right)^2\right)\frac{\partial^2\phi}{\partial y^2} - 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y} = 0$$

For isentropic flow (s = const.):

$$\left(\frac{\partial p}{\partial\rho}\right)_s = \frac{\mathrm{d}p}{\mathrm{d}\rho} = c^2 \quad\Longrightarrow\quad \nabla p = c^2\nabla\rho$$

<u>If</u> the flow is irrotational:

assumption to be checked

$$\nabla\times\boldsymbol{u} = 0$$

define a velocity potential

$$u = \frac{\partial\phi}{\partial x}, \qquad v = \frac{\partial\phi}{\partial y},$$

$$\nabla\times\boldsymbol{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) = 0$$

Is the flow irrotational?

conservation of momentum: $\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p$

Find a transport equation for vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$

$$\nabla \times | \quad \frac{\partial \boldsymbol{u}}{\partial t} + \nabla\left(\frac{|\boldsymbol{u}|^2}{2}\right) - \boldsymbol{u} \times \boldsymbol{\omega} = -\frac{1}{\rho}\nabla p$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} - \boldsymbol{\omega}\nabla \cdot \boldsymbol{u} + \frac{1}{\rho^2}\nabla\rho \times \nabla p$$

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} - \boldsymbol{\omega}\nabla \cdot \boldsymbol{u} + \cancelto{0}{\frac{c^2}{\rho^2}\nabla\rho \times \nabla\rho}$$

If $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = 0 \implies \frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{0}$

Indentities

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nabla\left(\frac{|\boldsymbol{u}|^2}{2}\right) - \boldsymbol{u} \times \boldsymbol{\omega}$$

$$\nabla \times (\nabla f) = 0$$

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) = \boldsymbol{u}\cancel{\nabla \cdot \boldsymbol{\omega}} - \boldsymbol{\omega}\nabla \cdot \boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} + (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}$$

$$\nabla \cdot (\nabla \times \boldsymbol{u}) = \nabla \cdot \boldsymbol{\omega} = 0$$

barotropic flow

$$\nabla p = c^2 \nabla\rho$$

If the incoming flow is irrotational, it remains irrotational (provided Re → ∞ and *s* = const.).

Speed of sound

Steady compressible potential flow (continuity + momentum eqs.):

$$(c^2 - \phi_x^2)\phi_{xx} + (c^2 - \phi_y^2)\phi_{yy} - 2\phi_x\phi_y\phi_{xy} = 0,$$

! 2 unknowns: $\phi, c \implies$ find eq. for c

Thermodynamic relationships

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \kappa R T = (\kappa - 1) c_p T$$

$$h = c_p T$$

$$h = \frac{c^2}{\kappa - 1}$$

Total energy equation

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(h + \frac{|\boldsymbol{u}|^2}{2}\right) = \cancel{\frac{\mathrm{D}p_A}{\mathrm{D}t} - \boldsymbol{u} \cdot \nabla p_A} + \cancel{\nabla \cdot (\underline{\underline{\tau}}\boldsymbol{u})} + \cancel{\rho \boldsymbol{g} \cdot \boldsymbol{u}} + \cancel{\nabla \cdot (k \nabla T)}$$

$\mathrm{Str} \to 0$ $\quad \frac{\cos(\boldsymbol{g} \measuredangle \boldsymbol{u})}{\mathrm{Fr}^2} \to 0$

$\mathrm{Re} \to \infty$ $\quad \mathrm{Pe} \to \infty$

$$h + \frac{|\boldsymbol{u}|^2}{2} = \text{const.}$$ along streamlines

$$\frac{c^2}{\kappa - 1} + \frac{|\boldsymbol{u}|^2}{2} = \text{const.} = \frac{c_\infty^2}{\kappa - 1} + \frac{U^2}{2}$$

$$c^2 = c_\infty^2 + \frac{\kappa - 1}{2}\left(U^2 - \phi_x^2 - \phi_y^2\right)$$