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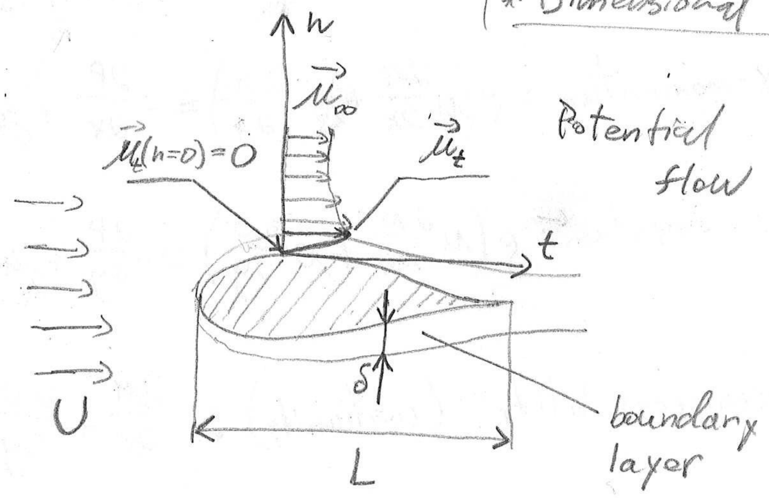
26.4.2024

| * Dimensional |

Boundary layer equations

Schlichting & Gersten (2017), pp. 145-146



No-slip condition

Empirical observation:

- the velocity drops to 0 at the surface, i.e., the velocity is continuous across the solid-liquid boundary
  the molecules of the fluid interact with the molecules of the solid
- for large Re, u_t drops to 0 across a thin boundary layer

However! The potential flow is uniquely determined by $u_n=0$ at the surface (flow tangency / no penetration).

$\Rightarrow$ Not possible to prescribe both, $u_n=0$ and $u_t=0$, for the potential flow

$\Rightarrow$ Potential flow solution valid only outside the B.L.

Why does the potential-flow theory fail to describe the boundary layer (and the no-slip condition)?

$\rightarrow$ Remember under which assumptions we have derived the potential flow equations:

- Start from Navier-Stokes equations for a steady incompressible flow (conservation of momentum and mass/volume) x-momentum: $\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$

y-momentum: $\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$

incompressibility (continuity): $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

Order-of-magnitude analysis: Potential flow (S&G, p. 147)

- estimate the characteristic magnitudes of the variables:
  $u \sim \Delta u \sim U_\infty$; $v \sim \Delta v \sim U_\infty$; $\Delta x \sim L$; $\Delta y \sim L$; $\Delta p \sim ?$
- estimate the magnitudes of all terms:

x-momentum: $\rho u\frac{\partial u}{\partial x} \sim \rho v\frac{\partial u}{\partial y} \sim \frac{\rho U_\infty^2}{L}$

$$\frac{\partial p}{\partial x} \sim \frac{\Delta p}{L}$$

$$\mu\frac{\partial^2 u}{\partial x^2} \sim \mu\frac{\partial^2 u}{\partial y^2} \sim \mu\frac{U_\infty}{L^2}$$

Observation: Pressure is essential for the kinematic BCs.
The pressure terms must compensate the convective term when the flow stagnates or deflects.

$$\Rightarrow \Delta p \sim \tfrac{1}{2}\rho U_\infty^2 \Rightarrow \frac{\partial p}{\partial x} \sim \frac{\rho U_\infty^2}{L} \sim \rho\frac{Du}{Dt}$$

Compare the magnitudes of the inertial and viscous terms:

$$\frac{\rho\, Du/Dt}{\mu \nabla^2 u} \sim \frac{\rho U_\infty^2}{L} \Big/ \frac{\mu U_\infty}{L^2} = \frac{\rho U_\infty L}{\mu} = Re \gg 1$$

$\Rightarrow$ viscous terms negligible

$\Rightarrow$ Euler equations $\Rightarrow$ Potential flow

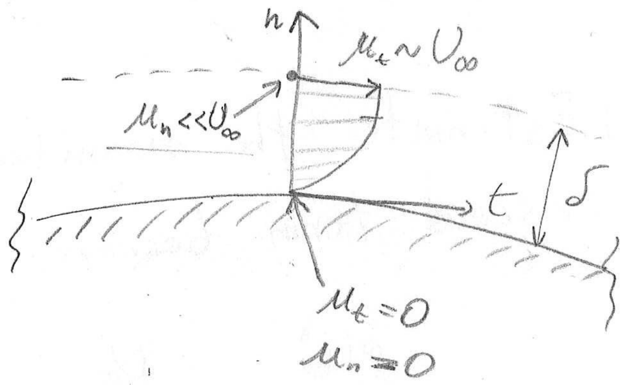
Boundary layer - different length-scales in wall-normal and wall-tangential directions !

$\rightarrow$ local coordinate system t-n

$\Delta t \sim L$, but $\Delta n \sim \delta$!

$u_t \sim \Delta u_t \sim U_\infty$

$u_n \sim \Delta u_n \ll U_\infty$!



(small surface curvature)

$$\frac{\partial u_t}{\partial t} \sim \frac{U_\infty}{L}; \quad \frac{\partial u_n}{\partial n} \sim \frac{\Delta u_n}{\delta};$$

Continuity equation: $\frac{\partial u_t}{\partial t} + \frac{\partial u_n}{\partial n} = 0 \Rightarrow \frac{\partial u_t}{\partial t} \sim \frac{\partial u_n}{\partial n}$

$$\frac{U_\infty}{L} \sim \frac{\Delta u_n}{\delta}$$

$$u_n \sim \Delta u_n \sim \frac{\delta U_\infty}{L} \ll U_\infty$$

Estimate magnitudes of different terms in the momentum equations $\rightarrow$ $\boxed{\text{In class task}}$ $\leftarrow$ which terms are important?

<u>t-momentum:</u> $\rho\left(u_t\frac{\partial u_t}{\partial t}+u_n\frac{\partial u_t}{\partial n}\right)=-\frac{\partial p}{\partial t}+\mu\left(\cancel{\frac{\partial^2 u_t}{\partial t^2}}+\frac{\partial^2 u_t}{\partial n^2}\right)$

Compare convective terms:

1) $\rho u_t\frac{\partial u_t}{\partial t}\sim\rho\frac{U_\infty^2}{L}$; $\rho u_n\frac{\partial u_t}{\partial n}\sim\rho\frac{\delta U_\infty}{L}\cdot\frac{U_\infty}{\delta}=\rho\frac{U_\infty^2}{L}\sim\rho u_t\frac{\partial u_t}{\partial t}$

$\mu\frac{\partial^2 u_t}{\partial t^2}\sim\mu\frac{U_\infty}{L^2}$; $\mu\frac{\partial^2 u_t}{\partial n^2}\sim\mu\frac{U_\infty}{\delta^2}$

2) <u>Compare the viscous terms:</u>

$$\frac{\mu\,\partial^2 u_t/\partial n^2}{\mu\,\partial^2 u_t/\partial t^2}\sim\frac{L^2}{\delta^2}\gg 1\Rightarrow\mu\frac{\partial^2 u_t}{\partial n^2}\gg\mu\frac{\partial^2 u_t}{\partial t^2}$$

3) Estimate the magnitude of $\frac{\delta}{L}$, such that the larger viscous term becomes comparable to the convective terms:

$$\frac{\rho U_\infty^{\cancel{2}}}{L}\sim\mu\frac{\cancel{U_\infty}}{\delta^2}\Rightarrow\frac{\delta^2}{L^2}\sim\frac{\mu}{\rho U_\infty L}=\frac{1}{Re}$$

$$\underline{\frac{\delta}{L}\sim\frac{1}{\sqrt{Re}}}\qquad(\text{S\&G, p. 146, Eq. 6.1})$$

The pressure term is again:

$$\frac{\partial p}{\partial t}\sim\rho\frac{U_\infty^2}{L}$$

<u>Side note:</u> There exists a viscous sub-layer where $\frac{n}{L}\ll\frac{1}{\sqrt{Re}}$ such that only the viscous terms are important

n-momentum:

$$\rho u_t \frac{\partial u_n}{\partial t} \sim \rho \frac{U_\infty^2 \delta}{L^2} \; ; \qquad \mu \frac{\partial^2 u_n}{\partial t^2} \sim \mu \frac{\delta U_\infty}{L^3}$$

$$\rho u_n \frac{\partial u_n}{\partial n} \sim \rho \frac{U_\infty^2 \delta}{L^2} \; ; \qquad \mu \frac{\partial^2 u_n}{\partial n^2} \sim \mu \frac{U_\infty}{L^2}$$

$$\cancel{\rho\left(u_t \frac{\partial u_n}{\partial t} + u_n \frac{\partial u_n}{\partial n}\right)} = -\frac{\partial p}{\partial n} + \cancel{\mu\left(\frac{\partial^2 u_n}{\partial t^2} + \frac{\partial^2 u_n}{\partial n^2}\right)}$$

$$\mu \frac{\partial^2 u_n}{\partial n^2} \gg \mu \frac{\partial^2 u_n}{\partial t^2}$$

$$\frac{\rho \, Du_n/Dt}{\mu \, \partial^2 u_n/\partial n^2} \sim \rho \frac{U_\infty^2 \delta}{\cancel{L^2}} \Big/ \mu \frac{U_\infty}{\cancel{L^2}} = \frac{\rho U_\infty \delta}{\mu} = Re \frac{\delta}{L} = \sqrt{Re} \gg 1$$

$$\frac{\partial p}{\partial n} \sim \frac{\rho U_\infty^2}{\delta} \; ; \qquad \frac{\rho \, Du_n/Dt}{\partial p/\partial n} \sim \cancel{\rho} \frac{U_\infty^2 \delta}{L^2} \Big/ \frac{\cancel{\rho} U_\infty^2}{\delta} = \frac{\delta^2}{L^2} \ll 1 \;!$$

n-momentum: $\boxed{\frac{\partial p}{\partial n} = 0}$! (assumption of small curvature)

⇒ pressure constant across the boundary layer
- → determined by the potential flow solution
- → classical attached laminar boundary layer does not affect the lift (at the leading order)

⇒ Computation of the flow in 2 steps:

1) Compute the potential flow with flow-tangency condition
2) Compute the boundary layer, using the known pressure from 1)
   - B.L. equations are parabolic!
   - outer pressure obtained from Bernoulli eq.

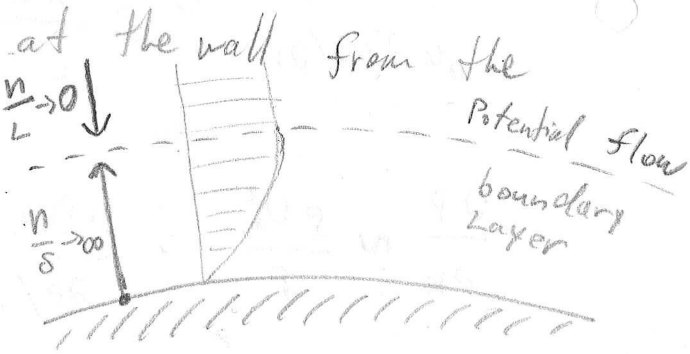
Boundary conditions for the boundary layer

- no-slip condition: $u_t = u_n = 0$ at $n = 0$
- matching with the outer flow (continuous velocity):

$$u_t \to U_e \quad \text{as } n/\delta \to \infty \quad \left(\text{i.e., } n \gg \frac{L}{\sqrt{Re}}\right)$$

where U_e is the tangential velocity at the wall from the potential flow solution:

$$U_e = \lim_{n/L \to 0} u_t$$



* "Van Dyke matching"

The boundary-layer solution provides the wall shear stress

$$\tau_w = \mu \frac{\partial u_t}{\partial n}\bigg|_{n=0}$$

Note: In addition to friction drag, B.L. also creates "form drag"
$\Rightarrow$ no stagnation pressure behind airfoil

$\Rightarrow$ skin-friction drag $\vec{F}_\tau = \oint \tau_w \, \vec{t} \; ds$

* In the outer flow, the Bernoulli equation holds:

$$\frac{1}{2}\rho u^2 + p = \text{const.} = \frac{1}{2}\rho u_\infty^2 + p_\infty$$

$$p - p_\infty = \frac{1}{2}\rho\left(u_\infty^2 - u^2\right) \quad / \nabla$$

$$\nabla p = -\frac{1}{2}\rho \nabla(u^2)$$

- In the boundary layer: $\frac{\partial p}{\partial n} = 0$

inside B.L.

$$-\frac{\partial p}{\partial t} = \rho U_e \frac{\partial U_e}{\partial t}$$

easily obtained from Euler eq.

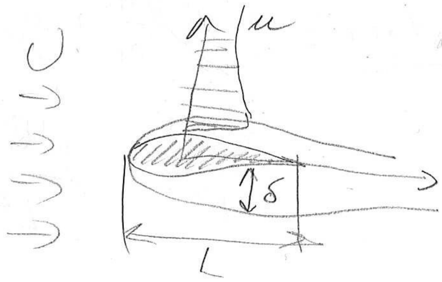
obtained from outer potential flow

Solution of B.L. eqs: $\Rightarrow$ Coordinate Transformation !

079 Laminar boundary layer

13.5.24

Dimensional



Recap: • Close to the surface there is a boundary layer of thickness δ

• Inside the B.L., viscosity becomes comparable to inertia

$\Rightarrow$ B.L. cannot be described by potential flow
(the flow is not irrotational inside B.L.)

$\Rightarrow$ describing B.L. requires a new system of eqs.

• B.L. needed to describe the no-slip condition
- velocity drops to 0 at the surface

B.L. is the cause of drag and separation

Deriving B.L. eqs.: - different length scale in wall-normal direction
$\Delta n \sim \delta$
- new scale for wall-normal velocity:
$\Delta u_n \sim ?$

1) Continuity equation:

$$\frac{\partial u_t}{\partial t} + \frac{\partial u_n}{\partial n} = 0$$

$$\frac{U_\infty}{L} \sim \frac{\Delta u_n}{\delta}$$

$$\Delta u_n \sim \frac{\delta U_\infty}{L}$$

2) t-momentum:

$$\rho\left(u_t \frac{\partial u_t}{\partial t} + u_n \frac{\partial u_t}{\partial n}\right) = -\frac{\partial p}{\partial t} + \mu\left(\cancel{\frac{\partial^2 u_t}{\partial t^2}} + \frac{\partial^2 u_t}{\partial n^2}\right)$$

$$\frac{\rho U_\infty^2}{L} \quad \frac{\rho U_\infty^2}{L} \quad \frac{\rho U_\infty^2}{L} \quad \frac{\mu U_\infty}{L^2} \ll \frac{\mu U_\infty}{\delta^2}$$

$$\frac{\mu U_\infty}{\delta^2} \sim \frac{\rho U_\infty^2}{L} \Rightarrow \frac{\delta^2}{L^2} \sim \frac{\mu}{\rho U_\infty L}$$

Recap

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<u>n-momentum</u>: $\rho\left(\cancel{u_t \frac{\partial u_n}{\partial t}} + \cancel{u_n \frac{\partial u_n}{\partial n}}\right) = -\frac{\partial p}{\partial n} + \mu\left(\cancel{\frac{\partial^2 u_n}{\partial t^2}} + \cancel{\frac{\partial^2 u_n}{\partial n^2}}\right)$

$$\frac{\rho U_\infty^2 \delta}{L^2} \qquad \frac{\rho U_\infty^2 \delta}{L^2} << \frac{\rho U_\infty^2}{\delta} \qquad \frac{\mu \delta U_\infty}{L^3} << \frac{\mu U_\infty}{L \delta^2}$$

$$\frac{\partial p}{\partial n} \Big/ \mu \frac{\partial^2 u_n}{\partial n^2} \sim \frac{\rho U_\infty^{\cancel{2}} L \delta^{\cancel{2}}}{\cancel{\delta} \mu \cancel{U_\infty}} = \frac{\rho U_\infty L}{\mu} \cdot \delta \sim \frac{Re}{\sqrt{Re}} = \underline{\underline{\sqrt{Re}}}$$

$$\frac{\partial p}{\partial n} >> \mu \frac{\partial^2 u_n}{\partial n^2}$$

$$\Rightarrow \boxed{\frac{\partial p}{\partial n} = 0}$$

(*) Note 2: Bernoulli does not hold inside B.L.

Note 1: Surface curvature was neglected !

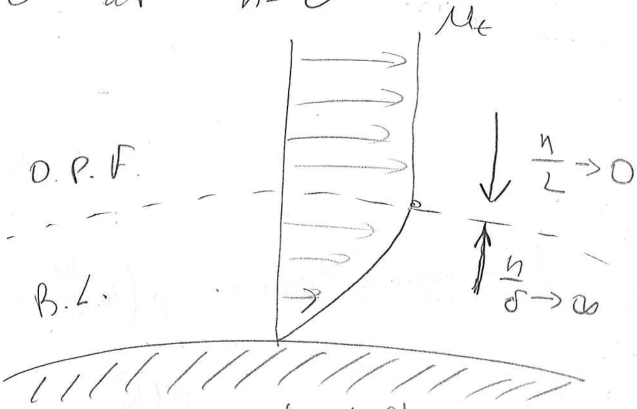
- Pressure across the boundary layer is constant (in n-direction)
- Pressure is continuous at the edge of B.L.
  $\Rightarrow$ "imposed" by the outer potential flow

$\Rightarrow$ <u>Computation workflow</u>: 1) Compute pressure from outer pot. flow (Bernoulli equation)
2) Solve B.L. equations with fixed pressure

<u>Boundary conditions</u>: - no-slip: $u_t = u_n = 0$ at $n = 0$

- matching with the outer flow

$$\lim_{\frac{n}{\delta} \to \infty} u_t^{(B.L.)} = \lim_{\frac{n}{L} \to 0} u_t^{(P.F.)}$$



<u>B.L. eq.</u>: $\rho\left(u_t \frac{\partial u_t}{\partial t} + u_n \frac{\partial u_t}{\partial n}\right) = -\frac{\partial p}{\partial t} + \mu \frac{\partial^2 u_t}{\partial n^2}$

tangential velocity at the wall from O.P.F.

(*) From momentum eq. in O.P.F.

$$p - p_\infty = \frac{1}{2}\rho\left(u_\infty^2 - u^2\right) \Rightarrow -\frac{\partial p}{\partial t} = \rho U_e \frac{\partial U_e}{\partial t} \; ; \; \cancel{U_e = u^{(O.P.F.)}}$$

Recap (R)

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Thus, the boundary layer equations can be written as follows in dimensional form

. t-momentum: $\rho\left(u_t \frac{\partial u_t}{\partial t} + u_n \frac{\partial u_t}{\partial n}\right) = \rho V_e \frac{\partial V_e}{\partial t} + \mu \frac{\partial^2 u_t}{\partial n^2}$

continuity: $\frac{\partial u_t}{\partial t} + \frac{\partial u_n}{\partial n} = 0$ (incompressible) (S&G, p. 153)

Remarks: (S&G, p. 148-149)

- B.L. eqs. are parabolic, that is, they describe an initial value problem in t-direction. Solution can be obtained by integration in t-direction, starting from given initial conditions.

  $\Rightarrow$ downstream conditions have no upstream influence

- The eqs. become invalid near separation points.

Change of variables

1) Dimensionless form in outer-flow scaling:

$$\tilde{u} = \frac{u}{u_\infty}, \quad \tilde{v} = \frac{v}{u_\infty}, \quad \tilde{U} = \frac{U}{u_\infty}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{y} = \frac{y}{L}$$

(S&G, p. 147)

$\rightarrow$ substitute into (7.4):

$$\frac{u_\infty^2}{L}\,\tilde{u}\,\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{u_\infty^2}{L}\,\tilde{v}\,\frac{\partial \tilde{u}}{\partial \tilde{y}} = \frac{u_\infty^2}{L}\,\tilde{U}\,\frac{d\tilde{U}}{d\tilde{x}} + \frac{\nu u_\infty}{L^2}\,\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \Big/ \cdot \frac{L}{u_\infty^2}$$

$$\tilde{u}\,\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\,\frac{\partial \tilde{u}}{\partial \tilde{y}} = \tilde{U}\,\frac{d\tilde{U}}{d\tilde{x}} + \frac{1}{Re}\,\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

⚠ However, $\tilde{v} \sim \frac{\tilde{u}}{\sqrt{Re}}$ and $\tilde{y} \sim \frac{\tilde{x}}{\sqrt{Re}}$ inside B.L.! ↯

Therefore, introduce a

1) Boundary-Layer Scaling:

(S&G, pp. 148-149)

$$\bar{u} = \frac{u}{u_\infty}, \quad \bar{v} = \frac{v}{u_\infty}\sqrt{Re}, \quad \bar{U} = \frac{U}{U_\infty}, \quad \bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}\sqrt{Re} \qquad (6.6)$$

$\rightarrow$ substitute into (7.4):

$$\boxed{\bar{u}\,\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\,\frac{\partial \bar{u}}{\partial \bar{y}} = \bar{U}\,\frac{d\bar{U}}{d\bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}} \qquad (6.14)$$

Note that the viscous term does not vanish in the boundary-layer scaling as $Re \rightarrow \infty$!

2) Stream function

- dimensional: $u = \frac{\partial \Psi}{\partial y}$, $v = -\frac{\partial \Psi}{\partial x}$ (S&G, pp. 154-155, Eq. 6.37)

=> continuity equation (7.5) is identically satisfied

- dimensionless, in boundary layer scaling:

$$\bar{\Psi} = \frac{\Psi}{L u_\infty}\sqrt{Re}, \quad \bar{u} = \frac{\partial \bar{\Psi}}{\partial \bar{y}} = \frac{\sqrt{Re}}{\cancel{L} u_\infty} \cdot \frac{\cancel{L}}{\cancel{\sqrt{Re}}} \frac{\partial \Psi}{\partial y} = \frac{u}{u_\infty}$$

$$\bar{v} = -\frac{\partial \bar{\Psi}}{\partial \bar{x}} = \frac{\sqrt{Re}}{\cancel{L} u_\infty} \cdot \cancel{L} \cdot \left(-\frac{\partial \Psi}{\partial x}\right) = \frac{v}{u_\infty}\sqrt{Re}$$

-> substituting into Eq. (6.14) we obtain:

$$\boxed{\frac{\partial \bar{\Psi}}{\partial \bar{y}} \frac{\partial^2 \bar{\Psi}}{\partial \bar{x} \partial \bar{y}} - \frac{\partial \bar{\Psi}}{\partial \bar{x}} \frac{\partial^2 \bar{\Psi}}{\partial \bar{y}^2} = \bar{U}\frac{d\bar{U}}{d\bar{x}} + \frac{\partial^3 \bar{\Psi}}{\partial \bar{y}^3}}$$

(For dimensional form, see Schlichting and Gersten, 2017, p. 155, Eq. 6.38)