Exercise 6 guidelines

[1] Let us consider a PDE on which a discretization has been applied on a domain $\Omega = [x_1, x_N]$. We assume for the sake of simplicity that the discretization is of the form

$$aT_{i-1}^{n+1} + bT_i^{n+1} + cT_{i+1}^{n+1} = dT_i^n, \tag{1}$$

where T is the unknown of the PDE, a, b, c, d some coefficients.

Boundary Condition implementation

Dirichlet Boundary Condition To implement a Dirichlet Boundary condition, one can (for example (several ways to do)) erase the corresponding line, set the diagonal value to 1, and the term on the right hand side to the Dirichlet BC value.

Neumann Boundary Condition To implement a Neumann Boundary condition on one boundary of the domain, one can introduce a so-called *ghost point* satisfying the condition

$$\frac{T_{N+1} - T_N}{\Delta x} = \beta, \tag{2}$$

where T_N is the value of the field T at x_N, T_{N+1} is the value of T at the ghost point and β the imposed flux. The condition (2) is the discretization of a Neumann boundary condition.

Injecting this condition in (1) leads to

$$aT_{N-1}^{n+1} + bT_N^{n+1} + c(T_N^{n+1} + \beta \Delta x) = dT_N^n. \tag{3}$$

For a homogeneous Neumann BC, this becomes

$$aT_{N-1}^{n+1} + (b + c)T_N^{n+1} = dT_N^n. \tag{4}$$

Example If we apply a Dirichlet BC at $x = x_1$ and a homogeneous Neumann BC at $x = x_N$ to the problem whose discretization is given by (1), the matrix-vector formulation becomes :

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & & & 0 \\ a & b & c & 0 & & & \\ 0 & a & b & c & 0 & & \vdots \\ & & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & & a & b & c \\ 0 & & \cdots & & 0 & a & b+c \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_{N-1} \\ T_N \end{pmatrix}^{n+1} = \begin{pmatrix} T_{Dirichlet} \\ d\,T_2^n \\ d\,T_3^n \\ \vdots \\ d\,T_{N-1}^n \\ d\,T_N^n \end{pmatrix}. \tag{5}$$

[1] More details and explanations can be found on goo.gl/e324sU and goo.gl/RY4yRx .