Fundamentals of Numerical Thermo-Fluid Dynamics 322.061

Exercise 2: Finite Difference Method

May 16, 2018

2.1) Determine the coefficients a to d in the formula:

$$\left[\frac{\partial u}{\partial x}\right]_j = au_{j-2} + bu_{j-1} + cu_{j+1} + du_{j+2}, \tag{1}$$

using Taylor series expansion and write down its relevant central difference approximation. It should be noted that the grid is uniform (Δx is constant). What is the truncation error of this formula?

2.2) Determine the coefficients a, b and c in the formula:

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_j = au_j + bu_{j+1} + cu_{j+2}, \tag{2}$$

and provide its relevant right-sided finite difference scheme. Which order does the scheme have?

2.3) For $y = \cos \pi x$ obtain du/dx at $x = 0.4$ with $\Delta x = 0.1$ analytically and using:

$$(a)\quad \frac{du}{dx} \approx \frac{u_{j+1} - u_j}{\Delta x},$$
$$(b)\quad \frac{du}{dx} \approx \frac{u_{j+1} - u_{j-1}}{2\Delta x},$$
$$(c)\quad \frac{du}{dx} \approx \frac{u_{j-2} - 8u_{j-1} + 8u_{j+1} - y_{j+2}}{12\Delta x}.$$

Then compare the results with the exact solution.

2.4) For $u = \cos \pi x$ obtain d^2u/dx^2 at $x = 0.4$ with $\Delta x = 0.1$, 0.05 and 0.025 using:

$$\frac{d^2u}{dx^2} \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2},$$

and compare the accuracy of the results.

2.5) Discretize the following equation using the scheme of 2.4).

$$\frac{\partial^2 u}{\partial x^2} = -\pi^2 cos(\pi\, x)$$

with $x \in [0, 4]$ with the Dirichlet Boundary conditions

$$u(x = 0) = 0$$
$$u(x = 1) = 0.$$

Write a program in MatLab that builds the matrix-vector system and solve it for $N = 10$ nodes, $N = 100$ nodes and $N = 1000$ nodes.

Note: The basics of finite difference approximations: `http://goo.gl/trlwYF`