Fundamentals of Numerical Thermo-Fluid Dynamics 322.061
Examples for home preparation

Exercise 2: Finite Difference Method

To be presented on May 22, 2019

2.1) Determine the coefficients a to d of the central difference scheme:

$$\left[\frac{\partial u}{\partial x}\right]_j = au_{j-2} + bu_{j-1} + cu_{j+1} + du_{j+2}, \tag{1}$$

using Taylor series expansion. What is the order of accuracy of the scheme? Is the scheme diffusive? Assume a uniformly spaced grid (Δx is constant).

2.2) For the function $u = \sin(\pi x)$ compute the first derivative du/dx at $x = 0.4$ using the schemes below with $\Delta x = 0.1$:

$$\begin{aligned}
(a)\quad & \frac{du}{dx} \approx \frac{u_{j+1} - u_j}{\Delta x}, \\
(b)\quad & \frac{du}{dx} \approx \frac{u_{j+1} - u_{j-1}}{2\Delta x}, \\
(c)\quad & \frac{du}{dx} \approx \frac{u_{j-2} - 8u_{j-1} + 8u_{j+1} - u_{j+2}}{12\Delta x}.
\end{aligned}$$

Then compare the results with the exact solution. How does the error evolve with the order of accuracy of the scheme?

2.3) For the function $u = \sin \pi x$ compute the second derivative d^2u/dx^2 at $x = 0.4$ with $\Delta x = 0.1$, 0.05 and 0.025 using the scheme:

$$\frac{d^2u}{dx^2} \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2},$$

Then compare the results with the exact solution. How does the error evolve with the grid spacing Δx?

2.4) The steady heat conduction in one dimension is governed by:

$$-\alpha \frac{\partial^2 T(x)}{\partial x^2} = q$$

where T is temperature, α is thermal conductivity and q is internal heat generation. With finite differences we can discretize the equation as

$$\frac{T_{j-1} - 2T_j + T_{j+1}}{\Delta x^2} = -\frac{q}{\alpha} \quad \text{for } j = 2, \ldots, N-1$$

Let us assume the case $q/\alpha = 1$ with fixed temperature on boundaries

$$T_1 = T_N = 0$$

Write the discrete problem in matrix-vector form

$$\mathbf{A}\,\vec{T} = \vec{b}$$

Then solve the problem using MatLab on a domain $\Omega = (0, 1)$ discretized into $N = 20$ uniformly spaced grid points.

Hint: Use the in-built function `spdiags` to create the matrix $\mathbf{A}$.