Fundamentals of Numerical Thermo-Fluid Dynamics 322.061
Examples for home preparation

Exercise 4: Explicit time-stepping schemes

To be presented on June 12, 2019

4.1 We want to solve the advection of passive scalar by time dependent velocity

$$\begin{cases} \partial_t u = -\cos(2\pi t)\partial_x u \quad , \text{on } \Omega = [0,1] \\ u(t=0,x) = e^{-\frac{(x-0.5)^2}{0.04}} \\ u(t,x=0) = u(t,x=1) \end{cases} \tag{1}$$

in one spatial dimension on a grid with uniform spacing Δx. Semi-discretization with second order central differences in space leads to an initial value problem

$$\partial_t \vec{u} = -\cos(2\pi t)\mathbf{A}\vec{u} \tag{2}$$

where $\mathbf{A}$ is the matrix form of the discrete first derivative.

Compute the solution at $t = 0.5$ in MatlLab, using the leap-frog scheme

$$\frac{\vec{u^{n+1}} - \vec{u^{n-1}}}{2\Delta t} = -\cos(2\pi t^n)\mathbf{A}\vec{u}^n \tag{3}$$

with $\Delta x = 0.01$, $\Delta t = 0.01$. Compare the result with the exact analytical solution and to the result of the first-order implicit Euler scheme. You may implement your code in the enclosed template function.

4.2 Solve the problem from the previous question with predictor-corrector scheme

$$\vec{u'} = \vec{u^n} - \Delta t \cos(2\pi t^n)\mathbf{A}\vec{u^n} \tag{4}$$

$$\vec{u^{n+1}} = \vec{u^n} - \frac{1}{2}\Delta t\mathbf{A}\left(\cos(2\pi t^n)\vec{u^n} + \cos(2\pi t^{n+1})\vec{u'}\right) \tag{5}$$

4.3 Compute the streamline of a steady rigid-body vortex

$$\dot{\vec{x}} \equiv \vec{u}(\vec{x}) = \begin{pmatrix} y \\ -x \end{pmatrix} \tag{6}$$

with second-order Runge-Kutta method, starting from the seeding point $\vec{x} = (1, 0)^T$ with time step $\Delta t = 0.5$. Then reduce the time step until you reach physical result.