Fundamentals of Numerical Thermo-Fluid Dynamics 322.061
Examples for home preparation

Exercise 3: Stability of Finite Difference schemes

To be presented on May 20, 2020

3.1) Let us consider a convection equation of passive scalar

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 0 \tag{1}$$

which we discretize with Leapfrog method:

$$\frac{T_j^{n+1} - T_j^{n-1}}{2\Delta t} = -u\frac{T_{j+1}^n - T_{j-1}^n}{2\Delta x} \tag{2}$$

(a) Determine the order of accuracy of the scheme in time and space

(b) Investigate the stability of this scheme using von Neumann analysis. Provide the conditions of stability if relevant.

3.2) Discretization of (1) with Lax-Friedrichs method reads

$$\frac{T_j^{n+1} - \frac{1}{2}(T_{j+1}^n + T_{j-1}^n)}{\Delta t} = -u\frac{T_{j+1}^n - T_{j-1}^n}{2\Delta x} \tag{3}$$

Show that (3) is equivalent to stabilizing forward-time central-space (FCTS) method by adding an artificial diffusion to the original equation:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \frac{(\Delta x)^2}{2\Delta t}\frac{\partial^2 u}{\partial x^2} \tag{4}$$

Hint: Compute the error of the approximation $u_j^n \approx \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)$

3.3) Implement the discretization (3) in MatLab in the form

$$\vec{u^{n+1}} = \mathbf{A}\vec{u^n} \tag{5}$$

Take $x \in [0;1], \Delta x = 0.05, \Delta t = 0.02, c = 1$. Apply the boundary condition

$$T_1^{n+1} = T_1^n \quad \text{at} \quad x_1 = 0$$

and the one-sided difference

$$T_N^{n+1} = \frac{c\Delta t}{\Delta x} T_{N-1}^n + \left(1 - \frac{c\Delta t}{\Delta x}\right) T_N^n \quad \text{at} \quad x_N = 1$$

Assuming an initial condition

$$T(x, t_0 = 0) = e^{-\frac{(x-0.5)^2}{0.08}}, \tag{6}$$

compute the solution at $t = 0.2$. Then show that one can solve equation (1) with the method of characteristics to obtain the exact solution

$$T(x, t) = e^{-\frac{(x-ct-0.5)^2}{0.08}} \tag{7}$$

and compare your numerical result with this analytical solution. Explain your observations.

3.4) Discretize a time-dependent convection-diffusion equation

$$\partial_t T + \partial_x T = \partial_{xx} T \quad \text{on } \Omega = [0; 1] \tag{8}$$

with any scheme from the table 6.18 of the lecture Skriptum.

(a) How many initial and boundary conditions do we need such that the problem has a unique solution?

(b) Implement the discretization in MatLab, using initial and boundary conditions of your choice

(c) Vary Δx and Δt to verify the stability boundaries of the scheme.