Fundamentals of Numerical Thermo-Fluid Dynamics 322.061
Examples for home preparation

Exercise 2: Finite Difference Method

To be presented on May 13, 2020

2.1 Determine the coefficients a to d of the central finite difference scheme for a uniformly spaced grid ($\Delta x = \text{const.}$):

$$\left[\frac{\partial u}{\partial x}\right]_j = au_{j-2} + bu_{j-1} + cu_{j+1} + du_{j+2}, \qquad (1)$$

using Taylor series expansion. What is the order of accuracy of the scheme? Is the scheme diffusive or dispersive?

Notation: $u_j \equiv u(x_j) \equiv u(j\Delta x)$

2.2 Determine the coefficients a and b of the following central finite difference scheme for a NON-UNIFORM grid ($\Delta x_j = x_{j+1} - x_j \neq \text{const.}$):

$$\left[\frac{\partial u}{\partial x}\right]_j = au_{j-1} + bu_{j+1}. \qquad (2)$$

What is the order of accuracy of this scheme? Would the order of accuracy change for a uniform spacing? Can you conclude any recommendations for designing a non-uniform finite-difference grids?

Notation: $x_{j-1} = x_j - \Delta x_{j-1}, \quad x_{j+1} = x_j + \Delta x_j$

2.3 For the function $u = \sin(\pi x)$ compute the first derivative du/dx at $x = 0.4$ using the schemes below:

$$(a) \quad \frac{du}{dx} \approx \frac{u_{j+1} - u_j}{\Delta x},$$
$$(b) \quad \frac{du}{dx} \approx \frac{u_{j+1} - u_{j-1}}{2\Delta x},$$
$$(c) \quad \frac{du}{dx} \approx \frac{u_{j-2} - 8u_{j-1} + 8u_{j+1} - u_{j+2}}{12\Delta x},$$

and compare the results with the exact solution. For each scheme, plot the dependence of the discretization error on Δx in logarithmic scale. Can you determine the order of accuracy of the scheme from the plot?

2.4 Consider the solved example *Solution of 1D steady diffusion*. Reduce the number of grid points N to check how many grid points you need to approximate the exact solution with a maximum relative error of 1%. Then change the source term to $f = 1$ and repeat. Explain your observation.

2.5 Consider the Friedrich's equation

$$\epsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} = a \quad \text{on } \Omega = [0, 1] \tag{3}$$

$$f = 0 \text{ at } x = 0 \tag{4}$$

$$f = 1 \text{ at } x = 1 \tag{5}$$

(a) Assume that f is a passive scalar (e.g. temperature or chemical concentration) transported in a fluid. Explain the physical meaning of each term of (3).

(b) Discretize (3) using finite differences for uniformly spaced grid.

(c) Solve the problem numerically for $\epsilon = 0.1$ and $a = 2$.

(d) What would be the main problem if ϵ gets too small? How would you approach such case?