Fundamentals of Numerical Thermo-Fluid Dynamics 322.061
Examples for home preparation

Exercise 1: Classification of Partial Differential Equations

To be submitted until May 6, 2020

1.1 A steady inviscid flow of a compressible fluid is described by the continuity equation

$$\nabla \cdot (\rho \vec{u}) = 0, \tag{1a}$$

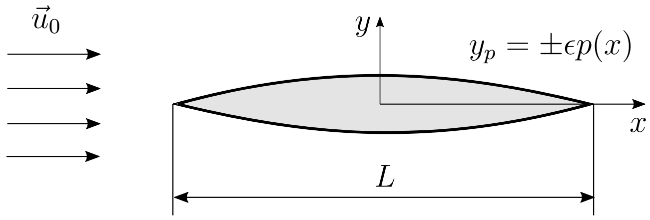
the momentum equation

$$(\vec{u} \cdot \vec{\nabla})\vec{u} = -\frac{c^2}{\rho}\vec{\nabla}\rho \tag{1b}$$

and the energy equation

$$\frac{c^2}{\kappa - 1} + \frac{1}{2}(u^2 + v^2) = h_T, \tag{1c}$$

where the unknowns $\rho(\vec{x}), \vec{u}(\vec{x}), c(\vec{x})$ are the density of the fluid, the velocity of the flow and the speed of sound respectively. The constants κ and h_T are the ratio of specific heats and the total entalphy of the flow. The system (1) is non-linear and cannot in general be solved analytically.



For a two-dimensional flow around a thin profile, $y_p = \pm\epsilon p(x)$, $\epsilon << 1$, the velocity field can be expanded[1] as

$$\vec{u} \sim \vec{u}_0 + \epsilon\vec{u}_1 + \mathcal{O}(\epsilon^2) \tag{2}$$

where $\vec{u}_0$ is the incoming flow (far away from the profile) and $\vec{u}_1$ is the leading order correction of the velocity field due to the presence of the profile. Scaling the lengths and velocities by L and $|\vec{u}_0|$ respectively, the equations for $\vec{u}_1 = (u_1, v_1)$ are

$$(1 - M_\infty^2)\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \tag{3a}$$

$$\frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y} = 0 \tag{3b}$$

where M_∞ is the Mach number.

(a) Determine the type (elliptic/parabolic/hyperbolic) of the system (3) depending on M_∞.

[1]more details in the LVA 322.074

(b) Using the velocity potential $u_1 = \partial\phi_1/\partial x$, $v_1 = \partial\phi_1/\partial y$, show that the system (3) can be converted into an equivalent second-order PDE

$$(1 - M_\infty^2)\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} = 0 \tag{4}$$

Is the type of the equation conserved?

1.2 (a) Find the characteristics of the equation (4) if they exist. Can one use the characteristics to find an analytical solution to the equation? If yes, how?

(b) The first order velocity field $\vec{u} = \vec{u}_0 + \epsilon\vec{u}_1$ from the previous question is tangential to the profile surface

$$\frac{v}{u} = \epsilon\frac{\partial p}{\partial x} \tag{5}$$

and the correction $\vec{u}_1$ should vanish away from the profile

$$\vec{u}_1 \to 0 \quad \text{for } x^2 + y^2 \to \infty. \tag{6}$$

Assume that the incoming flow is given by $\vec{u}_0 = (1, 0)$. Without solving the equation (3) or (4), try to sketch the first order velocity field $\vec{u}$ around the profile for $M_\infty < 1$ and $M_\infty > 1$ according to your educated expectation. What would be the difference between the two cases? Justify using the theory about characteristics.

1.3 (a) Determine the type of the time-dependent diffusion equation

$$\frac{\partial\phi}{\partial t} - \alpha\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = 0. \tag{7}$$

(b) Transform (7) into a system of first order equations and find its characteristics if they exist. Is the type of the problem conserved? Is the system equivalent to the original equation?

(c) Is the transformation unique? If not, does the choice of the transformation affect the resulting characteristics?

1.4 Assume that the time-dependence in (7) vanishes, $\partial\phi/\partial t = 0$.

(a) Repeat the same tasks from the previous example.

(b) Can you suggest an efficient and reliable way to analyse second-order PDEs with more than two independent variables using the method of characteristics?

1.5 Convert the Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{8}$$

into an equivalent system of first-order equations by introducing auxiliary unknowns and compute its characteristics. What is the physical meaning of the characteristics?

1.6 Create and present your own example to illustrate the physical consequences of the type of a PDE, and the meaning of characteristics.