Classification of PDEs

Consider an example of a partial differential equation

$$\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{i,j} \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu = f$$

The aim of this chapter is to introduce a classification which would describe the properties of the equation and its solution.

Basic classification

- Order
- Number of independent variables
- Linearity
- Constant/variable coefficients
- Homogenity

With this criteria we can describe our example as: *Linear, non-homogeneous* ($f \neq 0$) *, second-order PDE with* d *independent variables and constant coefficients* ($a_{i,j}, b_i, c = const.$).

In addition, second order PDEs and some systems of PDEs can be divided into three types: elliptic, parabolic and hyperbolic. The type of equation determines certain properties of the solution and it imposes restrictions on boundary conditions and discretization methods which can be used to solve it numerically. The next section illustrates each type on examples. It is followed by some methods to determine the type of an equation.

Properties of different types of PDEs

Elliptic

- Boundary conditions must be prescribed all along the boundary
- Local change of the boundary data influences the solution everywhere
- No propagation of waves
- No real characteristics

Parabolic

- Parabolic equations arise by adding time dependence $\frac{\partial u}{\partial t}$ to equation which was originally elliptic.
- Waves can propagate in one specific direction only, corresponding to time-like variable
- Equivalent system of n first-order equations has 1 to $n-1$ real characteristics
- Boundary conditions are required in space, initial condition in time
- Local change in boundary conditions affects immediately the solution at all positions in space, but only in future times

Hyperbolic

- Waves can reflect from boundaries
- Equivalent system of n first-order equations has n real characteristics.
- Boundary conditions are required in space, initial condition in time
- Information about local change of boundary condition travels through space with finite speed.

Types of boundary conditions

- Dirichlet: $u = u_D \quad \text{on } \Gamma_D$
- Newton: $\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma_N$
- Robin: $\frac{\partial u}{\partial n} + \alpha u = g_R \quad \text{on } \Gamma_R$
- periodic: $u(\Gamma_{P1}) = u(\Gamma_{P2})$

Methods to determine the type of PDE

Second order PDE

$$\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu = f$$

Second order PDEs describe a wide range of physical phenomena including fluid dynamics and heat transfer. It is convenient to classify them in terms of the coefficients multiplying the derivatives. Replacing $\frac{\partial}{\partial x_i}$ by s_i we can write the characteristic equation of the left hand side as

$$\sum_{i,j=1}^{d} s_i a_{i,j} s_j + \sum_{i=1}^{d} b_i s_i + c = 0$$

The PDE is:

1. **elliptic** if the matrix $a_{i,j}$ is (positive or negative) definite, i.e. its eigenvalues are non-zero and all have the same sign.
2. **hyperbolic** if the matrix $a_{i,j}$ has non-zero eigenvalues and all but one have the same sign.
3. **parabolic** if the matrix $a_{i,j}$ is semi-definite (i.e. exactly one eigenvalue is zero, while the others have the same sign) and $rank\,(a_{i,j}, b_i) = d$

Second order PDE with two independent variables

Note that in case of two independent variables we can write the second-order terms as

$$\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial u}{\partial x_j}\right) \equiv A\frac{\partial^2 u}{\partial x_1^2} + B\frac{\partial^2 u}{\partial x_1 \partial x_2} + C\frac{\partial^2 u}{\partial x_2^2}$$

The equation can then be classified as

- **elliptic** for $B^2 - 4AC < 0$
- **hyperbolic** for $B^2 - 4AC > 0$
- **parabolic** for $B^2 - 4AC = 0$

Caution: A and C must be non-zero for this method to work! If this is not the case, then use the generic method for second-order PDEs.

Remark: In case of variable coefficients, the equation can change its type in different parts of the domain.

Remark: In more than two independent variables, it is often useful to determine how the solution behaves on a certain two-dimensional sub-space. One can "freeze" $d-2$ independent variables on some fixed values (i.e. remove derivatives with respect to the "frozen" variables) and determine the type of the resulting reduced equation with two independent variables.

Systems of first-order PDEs with two independent variables

The general form can be written in Einstein's summation notation,

$$A_{i,j}\frac{\partial u_j}{\partial x} + B_{i,j}\frac{\partial u_j}{\partial y} + C_{i,j}u_j = f_i$$

or in matrix-vector form,

$$\mathbf{A}\partial_x\mathbf{u} + \mathbf{B}\partial_y\mathbf{u} + \mathbf{C}\mathbf{u} = \mathbf{f}$$

where $\mathbf{u}$ is the vector of unknowns. The characteristics can be computed from the matrices $\mathbf{A}$ and $\mathbf{B}$ by solving

$$\det\left(\mathbf{A}^t \mathrm{d}y - \mathbf{B}^t \mathrm{d}x\right) = 0$$

for either $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\frac{\mathrm{d}x}{\mathrm{d}y}$. We can conclude that the PDE is

- **elliptic** for no real solutions
- **hyperbolic** for n real solutions
- **parabolic** for 1 to $n-1$ real and no complex solutions

where n is the number of equations. Note that there are cases at which this method cannot determine the type of the system.

Systems of PDEs with three (or more) independent variables

$$A_{i,j}\frac{\partial u_j}{\partial x} + B_{i,j}\frac{\partial u_j}{\partial y} + C_{i,j}\frac{\partial u_j}{\partial z} = 0$$

The characteristics are always one dimension lower than the number of independent variables. That means that with three independent variables the characteristics are surfaces. The solution of the equation

$$\det\left(\mathbf{A}^t\lambda_x + \mathbf{B}^t\lambda_y + \mathbf{C}^t\lambda_z\right) = 0$$

now provides the normal vectors $\boldsymbol{\lambda} = (\lambda_x, \lambda_y, \lambda_z)$. One can divide the above equation by λ_x, λ_y or λ_z, provided it does not destroy any solutions. The classification is then the same as in the case of two independent variables.

Higher order PDEs

It is always possible to transform a higer order equation to a system of first-order equations by introducing auxiliary dependent variables. The methods above can then be used to investigate the type of the equation.

Method of characteristics

Characteristics of an equation with d independent variables are $d-1$ dimensional objects (curves for $d = 2$, surfaces for $d = 3$), such that the propagation of the solution along these objects can be described by ODE (i.e. partial derivatives can be replaced by total differentials). The number of real characteristics determines the type of an equation.

First order equations

Consider a first-order PDE with two independent variables

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = f$$

which we want to replace by an ODE

$$\frac{\mathrm{d}u}{\mathrm{d}t} = g$$

where the total derivative is defined as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial u}{\partial x}$$

on the characteristic curve $x = x(t)$. Dividing the original PDE by a we obtain the same form as the definition of the total derivative

$$\frac{f}{a} = \frac{\partial u}{\partial t} + \frac{b}{a}\frac{\partial u}{\partial x}$$

thus obtaining $\mathrm{d}u/\mathrm{d}t = f/a$ and $\mathrm{d}x/\mathrm{d}t = b/a$. After integration we obtain the characteristic

$$x(t) = x_0 + \frac{b}{a}t$$

along which the solution evolves as

$$u = u_0 + \frac{f}{a}t$$

The extension of this example to the case of multiple independent variables should be evident (see Appendix A). Finding the characteristics of higher order equations or systems of equations follows similar arguments.

Second order equations

Let us now consider the case of second-order PDE with two independent variables.

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = g$$

First we introduce auxiliary unknowns $P = \frac{\partial u}{\partial x}$, $Q = \frac{\partial u}{\partial y}$ with total differentials

$$\mathrm{d}P = \frac{\partial^2 u}{\partial x^2}\mathrm{d}x + \frac{\partial^2 u}{\partial x \partial y}\mathrm{d}y$$

$$\mathrm{d}Q = \frac{\partial^2 u}{\partial x \partial y}\mathrm{d}x + \frac{\partial^2 u}{\partial y^2}\mathrm{d}y$$

Note that this corresponds to converting the second-order equation into a system of first-order equations. Next we want to rewrite the original equation in terms of total derivatives $\frac{\mathrm{d}P}{\mathrm{d}x}, \frac{\mathrm{d}Q}{\mathrm{d}y}$. We try to expand

$$A\frac{\mathrm{d}P}{\mathrm{d}x} + C\frac{\mathrm{d}Q}{\mathrm{d}y} = A\frac{\partial^2 u}{\partial x^2} + \left(A\frac{\mathrm{d}y}{\mathrm{d}x} + C\frac{\mathrm{d}x}{\mathrm{d}y}\right)\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2}$$

and compare it to the original equation. We obtain

$$A\frac{\mathrm{d}y}{\mathrm{d}x} + C\frac{\mathrm{d}x}{\mathrm{d}y} = B$$

which after multiplication by $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\frac{\mathrm{d}x}{\mathrm{d}y}$, leads to

$$A\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - B\frac{\mathrm{d}y}{\mathrm{d}x} + C = 0$$

thus giving us the criteria

- $B^2 - 4AC < 0 \rightarrow$ no real characteristics $\rightarrow$ elliptic
- $B^2 - 4AC = 0 \rightarrow$ one real characteristic $\rightarrow$ parabolic

- $B^2 - 4AC > 0 \rightarrow$ two real characteristics $\rightarrow$ hyperbolic

Systems of first order equations with two independent variables

$$A_{i,j}\frac{\partial u_j}{\partial x} + B_{i,j}\frac{\partial u_j}{\partial y} + C_{i,j}u_j = f_i$$

Now we want to represent the linear combination of all lines of the system

$$L_i\left(A_{i,j}\frac{\partial u_j}{\partial x} + B_{i,j}\frac{\partial u_j}{\partial y} + C_{i,j}u_j\right) = L_i f_i$$

with linear combination of total differentials

$$\sum_i m_i \mathrm{d}u_i = \sum_i m_i \left(\frac{\partial u_i}{\partial x}\mathrm{d}x + \frac{\partial u_i}{\partial y}\mathrm{d}y\right)$$

Matching all partial derivatives we obtain

$$L_i\ A_{i,j} = m_j\ \mathrm{d}x$$

$$L_i\ B_{i,j} = m_j\ \mathrm{d}y$$

Multiplying the first set of equations by $\mathrm{d}x$ and the second set by $\mathrm{d}y$ and subtracting them leads to

$$(\mathbf{A}^T\mathrm{d}y - \mathbf{B}^T\mathrm{d}x)\mathbf{L} = \mathbf{0}$$

and from the solvability condition we obtain

$$\det\left(\mathbf{A}^T\mathrm{d}y - \mathbf{B}^T\mathrm{d}x\right) = 0$$

which can be solved for $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\frac{\mathrm{d}x}{\mathrm{d}y}$. Integration then leads to characteristics $y = y(x)$ or $x = x(y)$.

Examples

Second order PDE: Time-dependent diffusion

$$\frac{\partial u}{\partial t} - \Delta u = f$$

Considering one spatial variable x_1 and taking time as the second independent variable x_2, we can rewrite the equation as

$$\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} b_i\frac{\partial u}{\partial x_i} = f$$

with

```
a = [ 1 0
      0 0 ];
```

```
b = [ 0
      1 ];
eigs(a)
```

```
ans =
     0
     1
```

-

```
rank([a b])
```

```
ans = 2
```

As one eigenvalue is zero and the rank of the composite matrix $a|b$ equals the number of independent variables, we conclude that the equation is parabolic.

Task: Repeat this example for higher number of spatial dimensions.

Recommended literature

Lecture notes for the course LVA-Nr. 302.017: Grundlagen der numerischen Methoden der Stömungs- und Wärmetechnik, TU Wien

Otto, A. (2011), 'Classification of PDE's and Related Properties' In *Methods of Numerical Simulation in Fluids and Plasmas*. Lecture notes, University of Alaska

Fletcher, C. (1998), 'Partial Differential Equations' In *Computational Techniques for Fluid Dynamics 1: Fundamental and General Techniques,* Springer, Berlin.

Wesseling, P. (2001) 'Classification of partial differential equations' In Principles of Computational Fluid Dynamics, Vol. 29 of Springer Series in Computational Mathematics, Springer, Berlin.

Appendix A: Characteristics of first order PDE with multiple independent variables

Consider a generic first-order PDE

$$\sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu = f$$

which we want to replace by an ODE, e.g.

$$\frac{\mathrm{d}u}{\mathrm{d}x_1} + eu = g$$

where the total derivative is defined as

$$\frac{\mathrm{d}u}{\mathrm{d}x_1} = \frac{\partial u}{\partial x_1} + \frac{\mathrm{d}x_2}{\mathrm{d}x_1}\frac{\partial u}{\partial x_2} + \frac{\mathrm{d}x_3}{\mathrm{d}x_1}\frac{\partial u}{\partial x_3} + \dots$$

Rearranging the original PDE to the form

$$\frac{\partial u}{\partial x_1} + \sum_{i=2}^{d} \frac{b_i}{b_1}\frac{\partial u}{\partial x_i} + \frac{c}{b_1} = \frac{f}{b_1}$$

we obtain $\mathrm{d}x_i/\mathrm{d}x_1 = b_i/b_1$. Note that in case of $x_1 = t, x_2 = x, x_3 = y$ the characteristic is a curve evolving in time.