Shallow Water Channel

With the aid of a shallow water channel it is possible to visualize phenomena of gas dynamics. The theoretical foundation is called shallow water analogy, which states that under certain conditions a two-dimensional compressible gas flow and a flow of a liquid with a free surface are described by the same equations. Well known phenomena of gas dynamics like, e.g., Mach lines or shock fronts, can be visualized by a shallow water channel.

Various bodies are immersed into the supercritical flow and the flow pattern around the bodies is observed and interpreted.

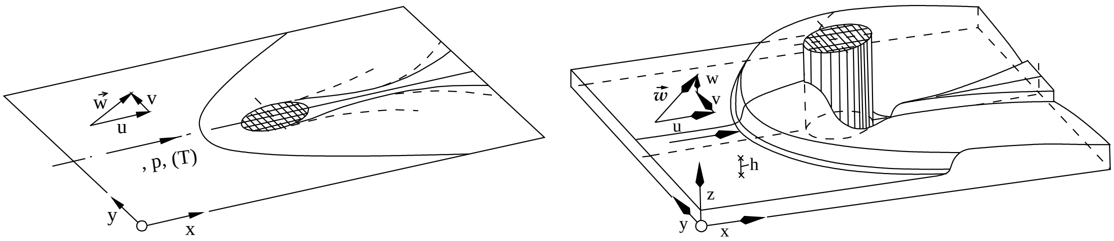

Figure 1: Comparison between two-dimensional, compressible gas flow (left) and shallow water flow (right). (from: Seminar in Flugantriebe, Karl Wörrlein, Fachgebiet Gasturbinen und Flugantriebe, TU Darmstadt, 2nd edition)

The shallow water channel basically consists of a flat plate, oriented almost horizontally, over which a thin film of water is flowing with a free surface. The transition from the subcritical to the supercritical flow state is accomplished by a Laval nozzle, i.e. by a narrowing and subsequent widening of the channel's side walls.

Derivation of the Analogy

In the following a brief outline of the derivation of the shallow water analogy is given. The flow is assumed to be frictionless. Therefore, e.g., boundary layers are neglected.

We choose the origin of the coordinate system at a place of the fluid at rest. The x-axis points into the direction of the main flow along the channel bottom. We neglect a possible small inclination of the channel bottom to the horizontal. The z-axis points vertically upwards, the y-axis is perpendicular to the drawing's plane. (see Fig. 2).

Bernoulli's equation along a streamline between the points P_0 $(0, y_0, z_0)$ and P (x, y, z)

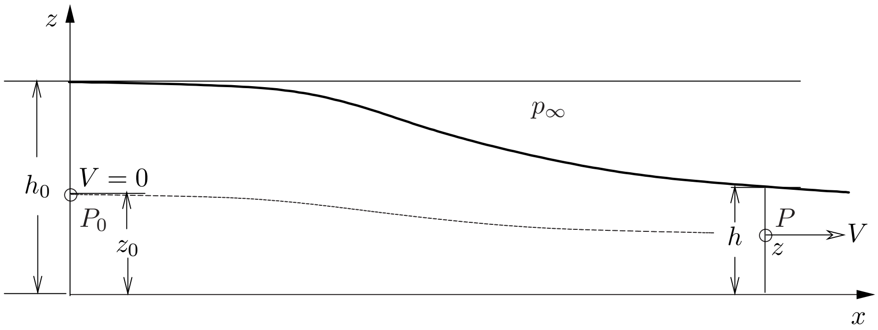

Figure 2: Definition of the quantities used.

reads

$$\frac{p_0}{\rho} + gz_0 = \frac{p}{\rho} + gz + \frac{V^2}{2}$$

where $V^2 \approx u^2 + v^2$ and u and v are the velocity components in x- and y-direction, respectively.

If we neglect the effect of the curvature of the streamlines on the pressure distribution, then we have a hydrostatic pressure distribution in every point of the water film with a free surface.

$$p = p_\infty + \rho g(h - z) \ .$$

If we insert this pressure into Bernoulli's equation, we get

$$gh_0 = gh + \frac{V^2}{2} \ .$$

Compared to the energy equation of gas dynamics,

$$c_p T_0 = c_p T + \frac{V^2}{2} \ .$$

we see that the liquid's depth in the shallow water channel corresponds to the temperature in the gas flow:

$$\boxed{\frac{h}{h_0} \quad \Longleftrightarrow \quad \frac{T}{T_0}}$$

In a similar way we compare the continuity equation of the water flow

$$\frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} = 0$$

with the corresponding equation for the gas flow

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \ .$$

The analogy between the depth of the water and the density

$$\boxed{\frac{h}{h_0} \quad \Longleftrightarrow \quad \frac{\rho}{\rho_0}}$$

is obvious.

For isentropic gas flow

$$\frac{T}{\rho^{\gamma-1}} = \text{const} \; .$$

holds. The analogous relation for the shallow water channel is

$$\frac{h}{h^{\gamma-1}} = \text{const} \; .$$

This requires a specific heat ratio of $\gamma = 2$. Such a kind of gas does not exist. This means that predictions for gas flow behaviour which were obtained from experiments in the shallow water channel are limited to cases where the value of the specific heat ratio γ is irrelevant. Therefore observations in the shallow water channel can be interpreted only qualitatively as gas flow behaviour, but not quantitatively.

A shock or a hydraulic jump, respectively, is no isentropic process. Subsequently, strictly speaking, of the relations derived, only the relation between the depth of the water and the density keeps to be valid for the comparison between a hydraulic jump and a shock.

From the energy and momentum equations for irrotational shallow water flow we get

$$(u^2 - gh)\frac{\partial u}{\partial x} + uv(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + (v^2 - gh)\frac{\partial v}{\partial y} = 0$$

A comparison with the corresponding equation from gas dynamics,

$$(u^2 - c^2)\frac{\partial u}{\partial x} + uv(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + (v^2 - c^2)\frac{\partial v}{\partial y} = 0$$

shows the analogy between the speed of sound and the depth of the water,

$$\boxed{c^2 \quad \Longleftrightarrow \quad gh}$$

$\sqrt{gh}$ is the propagation speed of gravitational waves in shallow water.

Finally, from the equation of state for a perfect gas

$$\frac{p}{p_0} = \frac{\rho T}{\rho_0 T_0}$$

one finds an analogous relation for the pressure

$$\boxed{\frac{p}{p_0} \quad \Longleftrightarrow \quad \frac{h^2}{h_0^2}} \; .$$

Summary

Gas flow, $\gamma = 2$		Shallow water flow	
relative temperature	T/T_0	relative water depth	h/h_0
relative density	ρ/ρ_0	relative water depth	h/h_0
relative pressure	p/p_0		h^2/h_0^2
speed of sound	c	wave propagation speed	$\sqrt{gh}$
Mach number	$\mathrm{M} = \frac{V}{c}$	Froude number	$\mathrm{Fr} = V/\sqrt{gh}$
subsonic flow	$\mathrm{M} < 1$	subcritical flow	$\mathrm{Fr} < 1$
supersonic flow	$\mathrm{M} > 1$	supercritical flow	$\mathrm{Fr} > 1$
shock		hydraulic jump	

Experiments

(1) Determining the Mach number

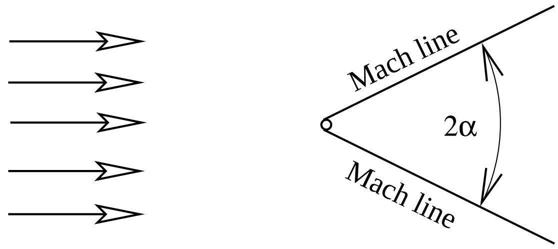

Figure 3: Definition of the Mach angle

Put a small disturbance (toothpick, needle, ...) into the flow within the supercritical area. Measure the angle α between the two waves propagating from the disturbance and calculate the equivalent to the Mach number (Froude number)

$$Fr = \frac{V}{\sqrt{gh}} = \frac{1}{\sin\alpha}$$

Determine the supercritical area in the flow by observing the propagation of disturbances (in supercritical flow disturbances do not propagate into the upstream direction).

(2) Flow around a wedge
Draw a sketch (ground view, side view) of the flow pattern around two peaked, wedge-shaped bodies and compare them to each other.

(3) Flow around a cylinder
Draw a sketch of the wave pattern around a circular cylinder. Compare with the flow around a wedge.

(4) Flow around an inclined plate
Draw a sketch of the flow around an inclined plate. Compare the different positions of the point of attack within the subcritical and the supercritical flow area.